Friedrich-Alexander-Universität
Faculty of Engineering

A Graded Monad for the Local Freshness Semantics of Nominal Automata with Name Allocation

Bachelor's Thesis in Computer Science

Hannes Schulze

Advisor: Prof. Dr. Lutz Schröder

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Hannes Schulze

Abstract

Nominal sets can be used to model data languages, which are languages over infinite alphabets. Regular nominal automata with name allocation (RNNA) first produce bar strings containing bound atoms that can be renamed under alpha-equivalence. Under local freshness semantics, the language generated by such automata contains all alpha-equivalent bar strings with the binders removed.

In this thesis, we present a coalgebraic definition for the trace semantics of RNNA in the framework of graded semantics. First, we define a general syntax and semantics for nominal algebra which can be used to define graded algebraic theories over nominal sets. It can then be shown that every such graded theory induces a graded monad over the category of nominal sets. We proceed provide an explicit description of a graded theory that can be used to capture exactly the trace semantics under local freshness.

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1 Introduction

Data languages, which are languages over infinite alphabets, are useful for modeling communication between processes where the range of values is infinite. As such, there is a desire to describe such languages using conservative extensions of regular automata for finite alphabets [NSV04]. *Nominal automata with name allocation* [Sch+21] provide such an extension, which is built around *nominal sets* [Pit13], a theory defining name abstraction and alpha-equivalence in an abstract setting.

These nominal automata introduce *binders* into the transitions of the automaton. A state may have *free transitions* and *bound transitions*. Bound transitions are closed under alpha-equivalence, thus introducing "placeholders" or "variables:" If there is a bound transition from one state to another, there are also transitions to all states that are alpha-equivalent to that successor state. Such automata can be constructed from a finite description by requiring that every state be finitely branching up to alpha-equivalence, even though there may be infinitely many transitions from any given state through alpha-equivalence. To describe their semantics, we first consider all words literally accepted by the automaton, including the binders. Then, under *local freshness semantics*, we consider all alpha-equivalent representatives of these words [Sch+21].

The goal of this thesis is to give a description of this semantics using *graded semantics* [MPS15]. This is a generic framework for describing the trace semantics of automata built around *universal coalgebra* [Rut00] – a generalization of state-based systems with transition relations. In systems such as labeled transition systems and regular automata, a family of transition relations specifies whether there is a transition from one state to another, where the states are elements of a set. Universal coalgebra extends this concept to arbitrary categories for states, defining the branching behavior of states as an endofunctor over that category. An instance of such a system is then given as a coalgebra for this functor. The graded semantics is based on encoding the *pretraces* of such systems, which consist of the traces and their respective poststates. These can be captured using *graded monads*, using grades to specify the depth of the pretraces; the unit corresponds to an empty pretrace for a state and multiplication corresponds to concatenating two pretraces. The graded semantics of a system can then be given using a natural transformation from the functor describing the system type into the graded monad. In the past, this framework has been used to formalize various equivalences for types of systems [For+25; FMS20] and multiple behavioral distances [DMS20].

To demonstrate the application of this framework to nominal automata with name allocation, we will first define a suitable notion of *graded theories* over nominal sets. These are algebraic equational theories that can be shown to induce graded monads in an abstract setting. This is inspired by similar formalizations for other categories [MPS15; For+25; FMS20]. We then give an explicit description of how the local freshness semantics of nominal automata can be formalized using this framework.

2 Preliminaries

We will first summarize some basic foundations of nominal sets, graded semantics, and nominal automata.

Throughout this thesis, we will fix a countably infinite set $\mathbb{A}$ of atoms.

2.1 Nominal Sets

We will start with some basic definitions and statements from nominal sets. For a more complete overview, see other sources [Pit13].

In the following, we let $\mathsf{Perm}(\mathbb{A})$ denote the group of permutations on $\mathbb{A}$; this is the set of bijective functions $\pi : \mathbb{A} \to \mathbb{A}$ such that $\pi(a) = a$ for almost all $a \in \mathbb{A}$. Recall that $\mathsf{Perm}(\mathbb{A})$ operates on a set X with a group action $\cdot : \mathsf{Perm}(\mathbb{A}) \times X \to X$ if $\mathsf{id} \cdot x = x$ and $\pi \cdot (\tau \cdot x) = (\pi \circ \tau) \cdot x$ for all $\pi, \tau \in \mathsf{Perm}(\mathbb{A})$ and $x \in X$. Given a set X equipped with such a group action $\cdot$, we will often write πx instead of $\pi \cdot x$. We recall some definitions regarding these permutation actions.

Definition 2.1 ([Pit13])**.** Let X, Y be sets equipped with permutation actions.

1. A function $f : X \to Y$ is **equivariant** if $f(\pi x) = \pi f(x)$ for all $x \in X$ and $\pi \in \mathsf{Perm}(\mathbb{A})$.
2. A relation $R \subseteq X \times X$ is **equivariant** if $x \mathrel{R} y$ implies $\pi x \mathrel{R} \pi y$ for all $x, y \in X$ and $\pi \in \mathsf{Perm}(\mathbb{A})$.
3. The **orbit** of $x \in X$ is $\mathsf{Perm}(\mathbb{A}) \cdot x \coloneqq \{\pi x : \pi \in \mathsf{Perm}(\mathbb{A})\}$.
4. If $x \in X$, then $\mathsf{fix}(x) \coloneqq \{\pi \in \mathsf{Perm}(\mathbb{A}) : \pi x = x\}$ is the set of permutations fixing x.
5. If $S \subseteq X$, then $\mathsf{Fix}(x) \coloneqq \bigcap_{x \in S} \mathsf{fix}(x)$ is the set of permutations fixing S.

This allows us to define a notion of variables upon which an element in a set "depends":

Definition 2.2 (Support [Pit13])**.** Let X be a set equipped with a permutation action. A set $S \subseteq \mathbb{A}$ **supports** an element $x \in X$ if $\mathsf{Fix}(S) \subseteq \mathsf{fix}(x)$.

Definition 2.3 (Nominal Sets [Pit13])**.** A set X equipped with a permutation action is a **nominal set** if every element $x \in X$ is finitely supported by some set $S \in \mathcal{P}_{\mathsf{f}}(\mathbb{A})$.

It easily follows that every element x in a nominal set has a least support, which we will denote $\mathsf{supp}(x)$. We will sometimes refer to it as the **support of** x. If an atom a does not occur in the support of x, we call it **fresh for** x and write $a \# x$.

Example 2.4. The set of λ terms, equipped with the permutation action which renames the variables in a term, forms a nominal set [Pit13]. In this context, $\mathsf{supp}(\lambda x. x\, y) = \{x, y\}$.

Since the support intuitively represents the names an element "depends" on, two permutations applied to an element yield the same results if the permutations are equal for all atoms in the support:

Proposition 2.5. *If* $\pi, \pi' \in \mathsf{Perm}(\mathbb{A})$ *and* $x \in X$ *with* $\pi(a) = \pi'(a)$ *for all* $a \in \mathsf{supp}(x)$*, then* $\pi x = \pi' x$*.*

Proof. For every $a \in \mathsf{supp}(x)$, we have $\pi(a) = \pi'(a)$ and thus $a = \pi^{-1}(\pi'(a))$. It follows that $\pi^{-1}\pi \in \mathsf{Fix}(\mathsf{supp}(x))$ and with this also $\pi^{-1}\pi' x = x$. □

The class of all nominal sets forms a category with equivariant functions as morphisms, which we will refer to as Nom.

Proposition 2.6. *If* $f : X \to Y$ *is an equivariant function between nominal sets, then* $\mathsf{supp}(f(x)) \subseteq \mathsf{supp}(x)$ *for all* $x \in X$*. If* f *is injective, then* $\mathsf{supp}(f(x)) = \mathsf{supp}(x)$*.*

Proof. Shown elsewhere [Pit13, Lemma 2.12]. □

Of particular interest will be the *name abstraction functor*:

Definition 2.7 (Name Abstraction Functor [Pit13])**.** Let the functor $[\mathbb{A}](-) : \mathsf{Nom} \to \mathsf{Nom}$ be defined by

$$[\mathbb{A}]X = (\mathbb{A} \times X)/{=_\alpha},$$
$$([\mathbb{A}]f)(\langle a\rangle x) = \langle a\rangle f(x),$$

where the relation $=_\alpha$ is defined as

$$(a, x) =_\alpha (a', x') \quad \Leftrightarrow \quad \exists c \in \mathbb{A} \setminus \mathsf{supp}(a, a', x, x').\ (a\ c)x = (a'\ c)x',$$

and the **name abstraction** $\langle a\rangle x$ is the equivalence class of (a, x).

Proposition 2.8. *If* $a, a' \in \mathbb{A}$ *and* $x, x' \in X$*, then* $\langle a\rangle x = \langle a'\rangle x'$ *iff one of the following statements is true:*

1. $a = a'$ *and* $x = x'$
2. $a \neq a'$*,* $a \# x'$*, and* $x = (a\ a')x'$*.*

Proof. Shown elsewhere [Pit13, Lemma 4.3]. □

Proposition 2.9. *Alpha-equivalence is an equivariant relation, in that for* $a, a' \in \mathbb{A}$*,* $x, x' \in X$*, and* $\pi \in \mathsf{Perm}(\mathbb{A})$*, we have*

$$\pi\langle a\rangle x = \langle \pi a\rangle \pi x = \langle \pi a'\rangle \pi x' = \pi\langle a'\rangle x' \quad \Leftrightarrow \quad \langle a\rangle x = \langle a'\rangle x'.$$

Proof. Shown elsewhere [Pit13, Section 4.2]. □

Given a nominal set X, we refer to the of finitely supported subsets as $\mathcal{P}_{\mathsf{fs}}(X)$. Equipped with element-wise permutation, this is itself a nominal set [Pit13]. Furthermore, we call a nominal set **orbit-finite** if the permutation action on it only has finitely many orbits.

Proposition 2.10. *The binary set operations* $\cup, \cap, \setminus$ *on finitely supported sets are equivariant.*

Proof. For $\cup$: Let $\pi \in \mathsf{Perm}(\mathbb{A}), L_1, L_2 \in \mathcal{P}_{\mathsf{fs}}(X)$. First, assume $\pi x \in \pi L_1 \cup \pi L_2$. W.l.o.g. assume that $\pi x \in \pi L_1$, and thus $x \in L_1 \subseteq L_1 \cup L_2$. Then it follows that $\pi x \in \pi(L_1 \cup L_2)$. Conversely, assume $\pi x \in \pi(L_1 \cup L_2)$, and thus $x \in L_1 \cup L_2$. W.l.o.g. assume that $x \in L_1$. Then it follows that $\pi x \in \pi L_1 \subseteq \pi L_1 \cup \pi L_2$.

For $\cap$ and $\setminus$: Shown elsewhere [Pit13, Proposition 1.9]. ☐

Proposition 2.11. *If $f : X \to Y$ is equivariant, then the direct image of finitely supported sets $f[\cdot] : \mathcal{P}_{\mathsf{fs}}(X) \to \mathcal{P}_{\mathsf{fs}}(Y)$ is also equivariant.*

Proof. Let $L \in \mathcal{P}_{\mathsf{fs}}(X)$ and $\pi \in \mathsf{Perm}(\mathbb{A})$. We have to show that $f[\pi L] = \pi f[L]$.

First assume that $f(\pi x) \in f[\pi L]$ with $\pi x \in \pi L$, where $x \in L$. By definition, we have $f(x) \in f[L]$. It follows from the equivariance of f that $f(\pi x) = \pi f(x) \in \pi f[L]$.

Conversely, assume that $\pi f(x) \in \pi f[L]$ with $f(x) \in f[L]$, where $x \in L$. By definition, we have $\pi x \in \pi L$ and, by equivariance of f, $\pi f(x) = f(\pi x) \in f[\pi L]$. ☐

The following are some useful properties of equivariant equivalence relations $\sim \subseteq X \times X$.

Proposition 2.12. *The canonical projection $[\cdot]_\sim : X \to X/{\sim}$ is equivariant: If $x \in X$ and $\pi \in \mathsf{Perm}(\mathbb{A})$, then $[\pi x]_\sim = \pi[x]_\sim$.*

Proof. First, assume that $y \in [\pi x]_\sim$; i.e., $y \sim \pi x$. Since $\sim$ is equivariant, it follows that $\pi^{-1} y \sim x$. Because of this, we get $\pi^{-1} y \in [x]_\sim$ and thus $y = \pi\pi^{-1} y \in \pi[x]_\sim$.

Conversely, assume that $\pi y \in \pi[x]_\sim$, with $y \in [x]_\sim$. This means that $y \sim x$. It follows from the equivariance of $\sim$ that $\pi y \sim \pi x$ and thus $\pi y \in [\pi x]_\sim$. ☐

Proposition 2.13. *A finite set of atoms $S \in \mathcal{P}_{\mathsf{f}}(\mathbb{A})$ supports an equivalence class $[x]_\sim \in X/{\sim}$ iff $\pi x \sim x$ for all $\pi \in \mathsf{Fix}(S)$.*

Proof. Let $\pi \in \mathsf{Fix}(S)$. It follows from Proposition 2.12 that $\pi[x]_\sim = [\pi x]_\sim$ and, by assumption, $[\pi x]_\sim = [x]_\sim$. ☐

Proposition 2.14. *If $[x]_\sim \in X/{\sim}$ is an equivalence class, then $\mathsf{supp}([x]_\sim) = \bigcap\{\mathsf{supp}(y) : y \in [x]_\sim\}$.*

Proof. Shown elsewhere [Pit13, Proposition 2.30]. ☐

Corollary 2.15. *If $x \in X$ is an equivalence class and $a \# [x]_\sim$, then there exists some $\tilde{x} \in X$ with $\tilde{x} \sim x$ and $a \# \tilde{x}$.*

Proof. Follows directly from Proposition 2.14 and the definition of $\bigcap$. ☐

2.2 Nominal Automata with Name Binding

With nominal sets as a foundation, we can recall how nominal automata with name allocation can be used to describe data languages [Sch+21].

We will start by defining words with binders in them:

Definition 2.16 (Extended Bar Alphabet and Bar Strings [Sch+21])**.** The **extended bar alphabet** $\bar{\mathbb{A}}$ is defined as

$$\bar{\mathbb{A}} = \mathbb{A} \cup \{|a : a \in \mathbb{A}\}.$$

We refer to words over $\bar{\mathbb{A}}$ as **bar strings**. Equipped with the letter-wise permutation action, the set of bar strings $\bar{\mathbb{A}}^\star$ is a nominal set.

Definition 2.17 (Alpha-Equivalence between Bar Strings [Sch+21])**.** We define alpha-equivalence on bar strings as the equivalence $\equiv_\alpha$ generated by

$$w|av \equiv_\alpha w|bu \quad \text{if } \langle a\rangle v = \langle b\rangle u \text{ in } [\mathbb{A}]\bar{\mathbb{A}}^\star.$$

We will denote the equivalence class of a word $w \in \bar{\mathbb{A}}^\star$ as $[w]_\alpha$.

Using name abstraction introduced in Definition 2.7, we can first define the requirements of regular nondeterministic nominal automata in the following way:

Definition 2.18 (RNNA [Sch+21])**.** A **regular nondeterministic nominal automaton (RNNA)** is a tuple $(A, \to, s, F)$ consisting of

- an orbit-finite set Q of **states**,
- an equivariant subset $\to \subseteq Q \times \bar{\mathbb{A}} \times Q$, called the **transition relation**,
- an **initial state** $s \in Q$,
- an equivariant subset $F \subseteq Q$ of **final states**,

such that

1. The relation $\to$ is α-invariant: If $s \xrightarrow{|a} q$ and $\langle a\rangle q = \langle a'\rangle q'$, then $s \xrightarrow{|a'} q'$.
2. The relation $\to$ is finitely branching up to α-equivalence: For every state $s \in Q$, the sets $\{(a, q) : s \xrightarrow{a} q\}$ and $\{\langle a\rangle q : s \xrightarrow{|a} q\}$ are finite.

The relation $\to$ is extended to words in the usual manner.

Since we will only consider trace semantics, we will always assume $F = Q$. We will often view RNNAs as orbit-finite coalgebras $\gamma : Q \to H(Q)$ for the functor H given by

$$H : \mathsf{Nom} \to \mathsf{Nom},$$
$$H(X) = \mathcal{P}_\mathsf{f}(\mathbb{A} \times X) \times \mathcal{P}_\mathsf{f}([\mathbb{A}]X).$$

Note that there are some significant changes from the original definition [Sch+21]: Since $F = Q$, there is no need to indicate whether a state is final. In addition, the power sets are not ufs, but finite – since we are only considering orbit-finite coalgebras, the two are equivalent [Sch+21, Lemma 2.2].

We will continue to describe the semantics of an RNNA:

Definition 2.19 (Literal and Bar Languages [Sch+21])**.** The **literal language** generated by a state $s \in Q$ of an RNNA given by $\gamma : Q \to H(Q)$ is defined as

$$L_0(s) = \{w \in \bar{\mathbb{A}}^\star : s \xrightarrow{w} q \text{ for some } q \in Q\}.$$

The **bar language** generated by s is the quotient

$$L_\alpha(s) = L_0(s)/{\equiv_\alpha}.$$

Definition 2.20 (Local Freshness Semantics [Sch+21])**.** The **local freshness semantics** of a bar language L is given by

$$D(L) = \{\mathsf{ub}(w) : [w]_\alpha \in L\}.$$

Example 2.21. Consider the RNNA given in Figure 2.1 with a countably infinite alphabet $\mathbb{A} = \{a, b, c, \ldots\}$.

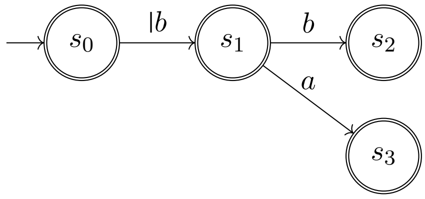

Figure 2.1: An RNNA, only showing one representative for alpha-equivalent transitions.

With this definition, we get

$$\begin{aligned} L_0(s_0) &= \{|bb, |ba, |cc, |ca, |dd, |da, \ldots\}, \\ L_\alpha(s_0) &= \{[|bb]_\alpha, [|ba]_\alpha\}, \\ D(L_\alpha(s_0)) &= \{aa, bb, ba, cc, ca, dd, da, \ldots\}. \end{aligned}$$

We will use the following statement, which provides information about the support of a successor state of a state:

Proposition 2.22. *Let $s, q \in Q$ be states of an RNNA given by $\gamma : Q \to H(Q)$ and $a \in \mathbb{A}$.*

1. *If $s \xrightarrow{a} q$, then* $\mathsf{supp}(q) \cup \{a\} \subseteq \mathsf{supp}(s)$*,*
2. *If $s \xrightarrow{|a} q$, then* $\mathsf{supp}(q) \subseteq \mathsf{supp}(s) \cup \{a\}$*.*

Proof. Shown elsewhere [Sch+21, Lemma 5.4]. □

2.3 Graded Semantics

Next, we will give a summary of graded semantics [MPS15].

First, recall the definition of graded monads:

Definition 2.23 (Graded Monads [MPS15])**.** A **graded monad** on a category $\mathbf{C}$ is a tuple $((M_n)_{n\in\mathbb{N}_0}, \eta, (\mu^{nk})_{n,k\in\mathbb{N}_0})$ containing

- for every $n \in \mathbb{N}_0$, an endofunctor $M_n : \mathbf{C} \to \mathbf{C}$,
- a **unit** transformation $\eta : \mathsf{Id} \to M_0$,
- for every $n, k \in \mathbb{N}_0$, a **multiplication** transformation $\mu^{nk} : M_n M_k \to M_{n+k}$,

satisfying

1. the **unit law**: $\forall n \in \mathbb{N}_0.\ \mu^{0,n} \cdot \eta M_n = \mathsf{id}_{M_n} = \mu^{n,0} \cdot M_n\eta$,
2. the **associative law**: $\forall n, k, m \in \mathbb{N}_0.\ \mu^{n,k+m} \cdot M_n\mu^{k,m} = \mu^{n+k,m} \cdot \mu^{n,k}M_m$.

Intuitively, $M_n(X)$ captures pretraces of length n with poststates from X. Given a functor describing the branching type of a system, we can then give a ”translation” from a branching step into the pretrace monad. This ”translation” is described by a graded semantics:

Definition 2.24 (Graded Semantics [MPS15])**.** A **graded semantics** for G-coalgebras consists of

- a graded monad $((M_n), \eta, (\mu^{nk}))$,
- a natural transformation $\alpha : G \to M_1$.

Given a G-coalgebra $\gamma : X \to G(X)$, the α**-pretrace sequence** is then given by

$$\gamma^{(0)} = (X \xrightarrow{\eta_X} M_0(X)),$$
$$\gamma^{(n+1)} = (X \xrightarrow{\alpha_X \circ \gamma} M_1(X) \xrightarrow{M_1(\gamma^{(n)})} M_1(M_n(X)) \xrightarrow{\mu_X^{1n}} M_{n+1}(X)),$$

and the α**-trace sequence** is defined as $(M_n(!) \circ \gamma^{(n)} : X \to M_n(1))_{n \in \mathbb{N}_0}$, where ! is the morphism into the terminal object of the category.

We call two G-coalgebras $\gamma : X \to G(X)$ and $\delta : Y \to G(Y)$ α**-trace equivalent** if $M_n(!) \circ \gamma^{(n)} = M_n(!) \circ \delta^{(n)}$ for all $n \in \mathbb{N}_0$.

By lifting the morphism to the terminal object into the graded monad, we "forget" about the poststates in the pretraces. This can be seen for labelled transition systems:

Example 2.25 (Labelled Transition Systems [MPS15])**.** We can view labelled transition systems as coalgebras for the functor $G : \mathsf{Set} \to \mathsf{Set}$ with

$$G(X) = \mathcal{P}(\mathbb{A} \times X),$$

where for every state, we give a set of transitions to other states.

The graded semantics with $M_n(X) = \mathcal{P}(\mathbb{A}^n \times X)$ and $\alpha = \mathsf{id}$ describes the trace-semantics of LTS: Let $\gamma : X \to G(X)$ describe the the LTS shown in Figure 2.2.

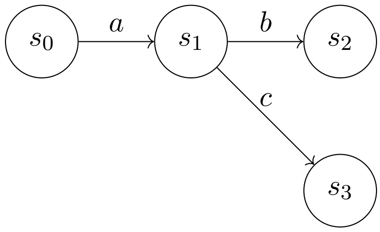

Figure 2.2: A depiction of an LTS.

At depth 2, we have

$$\gamma^{(2)}(s_0) = \{(ab, s_2), (ac, s_3)\},$$
$$(M_n(!) \circ \gamma^{(2)})(s_0) \cong \{ab, ac\},$$

matching exactly the traces of length 2.

On Set, it can be shown that every graded monad is induced by a **graded theory**, given by a (possibly class-sized) signature Σ, a class E of equational axioms, and a depth assignment $d(f) \in \mathbb{N}_0$ for every $f \in \Sigma$, such that all axioms have a uniform depth (i.e., variables have uniform depth 0 and $f(t_1, \ldots, t_k)$ with $d(f) = k$ has uniform depth $n + k$ if all t_i have uniform depth n) [MPS15]. In particular, every such theory induces a graded monad $((M_n), \eta, (\mu^{n,k}))$, where $M_n(X)$ consists of terms over X of uniform depth n modulo derivable equality, η converts variables into terms, and $\mu^{n,k}$ collapses layered terms, "removing" the inner equivalence classes [MPS15]. This motivates a similar construction for nominal sets.

There is an analogue of Eilenberg-Moore algebras for graded monads, namely graded algebras.

Definition 2.26 (Graded Algebras [MPS15])**.** Given a graded monad $((M_n), \eta, (\mu^{n,k}))$ on $\mathbf{C}$ and $n \in \mathbb{N}_0$, an M_n**-algebra** A consists of

- a family $(A_k)_{0 \leq k \leq n}$ of objects $A_k \in \mathsf{Ob}(\mathcal{C})$ called **carriers**, and
- a family $(a^{m,k})_{0 \leq m+k \leq n}$ of morphisms $a^{m,k} : M_m(A_k) \to A_{m+k}$,

satisfying

1. for $m \leq n$, $a^{0,m} \circ \eta_{A_m} = \mathsf{id}_{A_m}$,
2. for $m + r + k \leq n$, $a^{m,r+k} \circ M_m(a^{r,k}) = a^{m+r,k} \circ \mu^{m,r}_{A_k}$.

A **morphism between** M_n**-algebras** A, B is a family $(f_k)_{0 \leq k \leq n}$ of morphisms $f_k : A_k \to B_k$ such that, for $m + k \leq n$, $b^{m,k} \circ M_m(f_k) = f_{m+k} \circ a^{m,k}$.

Of particular interest are graded semantics built around depth-1 graded monads, which have compositionality properties useful for defining trace logics [MPS15, Section 8].

Definition 2.27 (Depth-1 Graded Monads [DMS20])**.** A graded monad $((M_n), \eta, (\mu^{nk}))$ is **depth-1** if the below diagram is a coequalizer in the category of M_0-algebras for all X and $n \in \mathbb{N}_0$:

$$M_1(M_0(M_n(X))) \underset{\mu^{1,0}_{M_n(X)}}{\overset{M_1(\mu^{0,n}_X)}{\rightrightarrows}} M_1(M_n(X)) \xrightarrow{\mu^{1,n}_X} M_{1+n}(X).$$

On Set, it can be shown that a graded monad is depth-1 iff all its operations and axioms have depth at most 1 [MPS15, Proposition 3.5].

3 Graded Theories over Nominal Sets

We will start by defining a concept of terms over nominal sets, which we will then use to define the notion of a graded theory over nominal sets. We will then show that every such graded theory induces a graded monad over Nom and that this graded monad is depth-1 if the operations and equations in the theory are all depth-1 (for a reasonable definition of depth).

The syntax and semantics defined in this chapter are inspired by previous work [Gab08], but with some notable differences: While Gabbay uses a plain set of variables and a set of freshness assertions as the context, we will just use a nominal set of variables and use that set as the "context". Furthermore, instead of defining a free atom as a standalone term and bound atoms as prefixes, we generalize this to operations taking a free or bound atom – this allows us to construct terms that are similar to regular bar expressions.

3.1 Syntax

Definition 3.1 (Graded Signatures)**.** A **graded signature** Σ is a tuple consisting of three disjoint sets:

- a set Σ_0 of **pure operations**,
- a set Σ_f of **free operations**, and
- a set Σ_b of **bound operations**,

along with an **arity** $\mathsf{ar}(f) \in \mathbb{N}_0$ and a **depth** $d(f) \in \mathbb{N}_0$ for each operation $f \in \Sigma_0 \cup \Sigma_\mathsf{f} \cup \Sigma_\mathsf{b}$.

We will write $f/n \in \Sigma_0$ if $f \in \Sigma_0$ and $\mathsf{ar}(f) = n$, and similar for Σ_f and Σ_b.

Definition 3.2 (Nominal Terms)**.** Given a nominal set X of **variables** and a graded signature Σ, define a **nominal Σ-term (over X)** as

$$t ::= x \mid f(t_1, \dots, t_p) \mid a.g(t_1, \dots, t_q) \mid \nu a.h(t_1, \dots, t_r),$$

ranging over all $x \in X$, $a \in \mathbb{A}$, $f/p \in \Sigma_0$, $g/q \in \Sigma_\mathsf{f}$, and $h/r \in \Sigma_\mathsf{b}$.

Definition 3.3 (Uniform Depth)**.** A nominal Σ-term t over a nominal set X of variables has **(uniform) depth** $n \in \mathbb{N}_0$ iff

- $t = x$ for $x \in X$ and $n = 0$.
- $t = f(t_1, \dots, t_p)$, where $t_1, \dots, t_p$ all have uniform depth n', and $n = n' + d(f)$.
- $t = a.f(t_1, \dots, t_p)$, where $t_1, \dots, t_p$ all have uniform depth n', and $n = n' + d(f)$.
- $t = \nu a.f(t_1, \dots, t_p)$, where $t_1, \dots, t_p$ all have uniform depth n', and $n = n' + d(f)$.

We define $\mathsf{Term}_{\Sigma,n}(X)$ to be the set of Σ-terms over variables X with uniform depth n.

Note that if an operation c has arity $\mathsf{ar}(c) = 0$ (meaning that it is a constant), it has uniform depth $n \in \mathbb{N}_0$ for all $n \geq d(c)$.

It follows directly from this definition that terms with uniform depth > 0 may never be just variables. However, a term with an operation may have uniform depth 0 if the operation is depth-0.

Definition 3.4 (Uniform Substitution)**.** Given nominal sets of variables X, Y, a depth $l \in \mathbb{N}_0$, and a graded signature Σ, a **(uniform) substitution** is a function $\sigma : Y \to \mathsf{Term}_{\Sigma,l}(X)$.

Its **application** $(\cdot)\sigma : \mathsf{Term}_{\Sigma,m}(Y) \to \mathsf{Term}_{\Sigma,m+l}(X)$ is recursively defined as

$$\begin{aligned} x\sigma &= \sigma(x), \\ (f(t_1, \ldots, t_p))\sigma &= f(t_1\sigma, \ldots, t_p\sigma), \\ (a.f(t_1, \ldots, t_p))\sigma &= a.f(t_1\sigma, \ldots, t_p\sigma), \\ (\nu a.f(t_1, \ldots, t_p))\sigma &= \nu a.f(t_1\sigma, \ldots, t_p\sigma). \end{aligned}$$

Definition 3.5 (Permutation Action on terms)**.** Given a nominal set X of variables, a depth $m \in \mathbb{N}_0$, and a graded signature Σ, we recursively define a permutation action for $\pi \in \mathsf{Perm}(\mathbb{A})$ on $\mathsf{Term}_{\Sigma,m}(X)$ with

$$\begin{aligned} \pi x &= \pi \star x, \\ \pi(f(t_1, \ldots, t_p)) &= f(\pi t_1, \ldots, \pi t_p), \\ \pi(a.f(t_1, \ldots, t_p)) &= \pi(a).f(\pi t_1, \ldots, \pi t_p), \\ \pi(\nu a.f(t_1, \ldots, t_p)) &= \nu\pi(a).f(\pi t_1, \ldots, \pi t_p), \end{aligned}$$

where $\star$ denotes the permutation action on the nominal set X. Equipped with this permutation action, $\mathsf{Term}_{\Sigma,n}(X)$ is a nominal set.

3.2 A Derivation System

We proceed to define the concept of a graded theory for nominal sets similar to prior works for other categories [FMS20].

Definition 3.6 (Equations)**.** Given a nominal set X of variables, a **depth-n Σ-equation (over X)** is a pair $(t, u) \in \mathsf{Term}_{\Sigma,n}(X)^2$, denoted as $X \vdash_n t = u$. When the depth of (t, u) is not important, we often speak of **Σ-equations**.

Definition 3.7 (Graded Theories)**.** A **graded theory** $T = (\Sigma, E)$ consists of a graded signature Σ and a class of Σ-equations E, referred to as **axioms**.

Definition 3.8 (Derivations)**.** Given a graded theory $T = (\Sigma, E)$, let **derivable equations** be inductively defined by Figure 3.1.

The $(\mathrm{ax}_{r=s})$ rules also include that we implicitly close the class of axioms under permutations. We furthermore require that the substitution σ is "derivably equivariant" (a notion formalized below). This is to ensure that derivable equality is an equivariant relation.

Definition 3.9 (Derivable Equivariance)**.** Given a graded theory $T = (\Sigma, E)$, a uniform substitution $\sigma : Y \to \mathsf{Term}_{\Sigma,l}(X)$ is **derivably equivariant** if

$$X \vdash_l \pi\sigma(x) = \sigma(\pi x)$$

is derivable for every $x \in Y$ and every $\pi \in \mathsf{Perm}(\mathbb{A})$.

$$\text{(refl)}\ \frac{}{X \vdash_0 x = x} \qquad \text{(symm)}\ \frac{X \vdash_m u = t}{X \vdash_m t = u} \qquad \text{(trans)}\ \frac{X \vdash_m t = v \qquad X \vdash_m v = u}{X \vdash_m t = u}$$

$$(\text{cong}_f)\ \frac{X \vdash_m t_i = u_i \qquad \forall i \in \{1, \dots, p\}}{X \vdash_{m+d(f)} f(t_1, \dots, t_p) = f(u_1, \dots, u_p)} \quad (f/p \in \Sigma_0)$$

$$(\text{cong}_{a.f})\ \frac{X \vdash_m t_i = u_i \qquad \forall i \in \{1, \dots, p\}}{X \vdash_{m+d(f)} a.f(t_1, \dots, t_p) = a.f(u_1, \dots, u_p)} \quad (f/p \in \Sigma_{\mathsf{f}}, a \in \mathbb{A})$$

$$(\text{cong}_{\nu a.f})\ \frac{X \vdash_m t_i = u_i \qquad \forall i \in \{1, \dots, p\}}{X \vdash_{m+d(f)} \nu a.f(t_1, \dots, t_p) = \nu a.f(u_1, \dots, u_p)} \quad (f/p \in \Sigma_{\mathsf{b}}, a \in \mathbb{A})$$

$$(\text{ax}_{r=s})\ \frac{X \vdash_l \pi\sigma(x) = \sigma(\pi x) \qquad \forall \pi \in \mathsf{Perm}(\mathbb{A}), x \in Y}{X \vdash_{m+l} (\tau r)\sigma = (\tau s)\sigma} \quad (Y \vdash_m r = s \in E, \tau \in \mathsf{Perm}(\mathbb{A}))$$

$$(\text{perm}_f)\ \frac{a\#u_i \qquad X \vdash_m t_i = (a\ b)u_i \qquad \forall i \in \{1, \dots, p\}}{X \vdash_{m+d(f)} \nu a.f(t_1, \dots, t_p) = \nu b.f(u_1, \dots, u_p)} \quad (f/p \in \Sigma_{\mathsf{b}}, a, b \in \mathbb{A}, a \neq b)$$

Figure 3.1: System of rules for deriving equations for a graded theory $T = (\Sigma, E)$.

Lemma 3.10. *Let $T = (\Sigma, E)$ be graded theory. If $\sigma : Y \to \mathsf{Term}_{\Sigma,l}(X)$ is derivably equivariant, then $X \vdash_{m+l} \pi(t\sigma) = (\pi t)\sigma$ can be derived for every $t \in \mathsf{Term}_{\Sigma,m}(X)$ and $\pi \in \mathsf{Perm}(\mathbb{A})$.*

Proof. By induction on t.

- *For $t = x$ with $x \in X$:* Since $\pi(t\sigma) = \pi\sigma(x)$ and $(\pi t)\sigma = \sigma(\pi x)$, it follows directly from the assumption that $X \vdash_l \pi(t\sigma) = (\pi t)\sigma$ is derivable.
- *For $t = f(t_1, \dots, t_p)$ with $f/p \in \Sigma_0$, $t_1, \dots, t_p \in \mathsf{Term}_{\Sigma,m'}(X)$, and $m = m' + d(f)$:* By inductive hypothesis, we can derive $X \vdash_{m'} \pi(t_i\sigma) = (\pi t_i)\sigma$ for every $i \in \{1, \dots, p\}$. By applying (cong$_f$), we get $X \vdash_m f(\pi(t_1\sigma), \dots, \pi(t_p\sigma)) = f((\pi t_1)\sigma, \dots, (\pi t_p)\sigma)$. Finally, we know that

$$\begin{aligned} \pi(t\sigma) &= \pi(f(t_1, \dots, t_p)\sigma) \\ &= \pi(f(t_1\sigma, \dots, t_p\sigma)) && \text{by Definition 3.4} \\ &= f(\pi(t_1\sigma), \dots, \pi(t_p\sigma)) && \text{by Definition 3.5,} \end{aligned}$$

and

$$\begin{aligned} (\pi t)\sigma &= (\pi(f(t_1, \dots, t_p)))\sigma \\ &= (f(\pi t_1, \dots, \pi t_p))\sigma && \text{by Definition 3.5} \\ &= f((\pi t_1)\sigma, \dots, (\pi t_p)\sigma) && \text{by Definition 3.4,} \end{aligned}$$

proving the statement.

- *For $t = a.f(t_1, \dots, t_p)$ and $t = \nu a.f(t_1, \dots, t_p)$:* Analogous to the above case.

□

Proposition 3.11. *Let $T = (\Sigma, E)$ be a graded theory and $t, u \in \mathsf{Term}_{\Sigma,m}(X)$. If $X \vdash_m t = u$ is derivable, then $X \vdash_m \pi t = \pi u$ is derivable for every $\pi \in \mathsf{Perm}(\mathbb{A})$.*

Proof. By induction on the derivation of $X \vdash_m t = u$.

- *For (refl):* We know that $t = u = x \in X$ and $m = 0$, and thus $\pi t = \pi x = \pi u$. It follows from (refl) that $X \vdash_0 \pi t = \pi u$.
- *For (symm):* We know that $X \vdash_m u = t$ is derivable and thus, by inductive hypothesis, $X \vdash_m \pi u = \pi t$. It follows from (symm) that $X \vdash_m \pi t = \pi u$.
- *For (trans):* We know that, for some v, $X \vdash_m t = v$ and $X \vdash_m v = u$ are derivable. By inductive hypothesis, it follows that $X \vdash_m \pi t = tv$ and $X \vdash_m \pi v = \pi u$. It follows from (trans) that $X \vdash_m \pi t = \pi u$.
- *For (cong$_f$):* We know that $t = f(t_1, \ldots, t_p)$, $u = f(u_1, \ldots, u_p)$, and $X \vdash_{m'} t_i = u_i$ is derivable for all $i \in \{1, \ldots, p\}$ with $m = m' + d(f)$. By inductive hypothesis, it follows that $X \vdash_{m'} \pi t_i = \pi u_i$ for all $i \in \{1, \ldots, p\}$. It follows from (cong$_f$) that $X \vdash_m f(\pi t_1, \ldots, \pi t_p) = f(\pi u_1, \ldots, \pi u_p)$. Finally, by applying Definition 3.5, we get $X \vdash_m \pi(f(t_1, \ldots, t_p)) = \pi(f(u_1, \ldots, u_p))$.
- *For (cong$_{a.f}$):* We know that $t = a.f(t_1, \ldots, t_p)$, $u = a.f(u_1, \ldots u_p)$, and $X \vdash_{m'} t_i = u_i$ is derivable for all $i \in \{1, \ldots, p\}$ with $m = m' + d(f)$. By inductive hypothesis, it follows that $X \vdash_{m'} \pi t_i = \pi u_i$ for all $i \in \{1, \ldots, p\}$. It follows from (cong$_{\pi(a).f}$) that $X \vdash_m \pi(a).f(\pi t_1, \ldots, \pi t_p) = \pi(a).f(\pi u_1, \ldots, \pi u_p)$. Finally, by applying Definition 3.5, we get $X \vdash_m \pi(a.f(t_1, \ldots, t_p)) = \pi(a.f(u_1, \ldots, u_p))$.
- *For (cong$_{\nu a.f}$):* Analogous to the above case.
- *For (ax$_{r=s}$):* We know that $Y \vdash_{m'} r = s \in E$ and that $t = (\tau r)\sigma, u = (\tau s)\sigma$ for a derivably equivariant substitution $\sigma : Y \to \mathsf{Term}_{\Sigma,l}(X)$ with $m = m' + l$ and a permutation $\tau \in \mathsf{Perm}(\mathbb{A})$. By applying (ax$_{r=s}$) with the permutation $\pi\tau$, we get $Y \vdash_m (\pi\tau r)\sigma = (\pi\tau s)\sigma$. Since, by assumption, σ is derivably equivariant, it follows from Lemma 3.10 that $Y \vdash_m \pi((\tau r)\sigma) = \pi((\tau s)\sigma)$ is derivable.
- *For (perm$_f$):* We know that $t = \nu a.f(t_1, \ldots, t_p)$ and $u = \nu b.f(u_1, \ldots, u_p)$, where $a\#u_i$ and $X \vdash_{m'} t_i = u_i$ is derivable for each $i \in \{1, \ldots, p\}$ with $m = m' + d(f)$. Furthermore, $a \neq b$. By inductive hypothesis, we know that $X \vdash_{m'} \pi t_i = \pi(a\ b)u_i$ is derivable for each i, where $\pi(a\ b)u_i = (\pi(a)\ \pi(b))(\pi u_i)$. Since π is a bijection, $\pi(a) \neq \pi(b)$. In addition, since supp is equivariant, it follows that $\pi(a)\#\pi u_i$. Thus, we can apply (perm$_f$) and get $X \vdash_m \nu\pi(a).f(t_1, \ldots, t_p) = \nu\pi(b).f(u_1, \ldots, u_p)$.

□

3.3 Nominal Algebras and Models

Definition 3.12 (Nominal Algebras)**.** Given a graded signature Σ and a depth $n \leq \omega$, a **nominal** (Σ, n)**-algebra** A consists of

- a family $(A_i)_{0 \leq i \leq n}$ of nominal sets, called **carriers**,
- for $f/p \in \Sigma_0$ with $m \in \mathbb{N}_0, m + d(f) \leq n$, an equivariant function $f_{A,m} : A_m^p \to A_{m+d(f)}$,
- for $f/p \in \Sigma_{\mathsf{f}}$ with $m \in \mathbb{N}_0, m+d(f) \leq n$, an equivariant function $f_{A,m} : \mathbb{A} \times A_m^p \to A_{m+d(f)}$,
- for $f/p \in \Sigma_{\mathsf{b}}$ with $m \in \mathbb{N}_0, m+d(f) \leq n$, an equivariant function $f_{A,m} : [\mathbb{A}]A_m^p \to A_{m+d(f)}$.

A **morphism between** (Σ, n)**-algebras** A, B is a family $(h_i)_{0 \leq i \leq n}$ of equivariant functions

$h_i : A_i \to B_i$ satisfying the following properties:

1. For $f/p \in \Sigma_0$ and $x_1, \ldots, x_p \in A_m$,
$$h_{m+d(f)}(f_{A,m}(x_1, \ldots, x_p)) = f_{B,m}(h_m(x_1), \ldots, h_m(x_p)).$$
2. For $f/p \in \Sigma_\mathsf{f}$, $a \in \mathbb{A}$, and $x_1, \ldots, x_p \in A_m$,
$$h_{m+d(f)}(a.f_{A,m}(x_1, \ldots, x_p)) = a.f_{B,m}(h_m(x_1), \ldots, h_m(x_p).)$$
3. For $f/p \in \Sigma_\mathsf{b}$, $a \in \mathbb{A}$, and $x_1, \ldots, x_p \in A_m$,
$$h_{m+d(f)}(\nu a.f_{A,m}(x_1, \ldots, x_p)) = \nu a.f_{B,m}(h_m(x_1), \ldots, h_m(x_p)).$$

We define $\mathsf{Alg}(\Sigma, n)$ to be the category of nominal (Σ, n)-algebras and the morphisms between them.

Intuitively, one can think of the elements in A_i as terms evaluated up to depth i.

Definition 3.13 (Evaluation Map)**.** Given a nominal (Σ, n)-algebra A, a base depth $k \in \mathbb{N}_0, k \leq n$, and an equivariant **environment** $\iota : X \to A_k$, the **evaluation map** $[\![\cdot]\!]^\iota_m : \mathsf{Term}_{\Sigma,m}(X) \to A_{k+m}$ of depth-m terms with $k + m \leq n$ is defined recursively by

$$\begin{aligned}
[\![x]\!]^\iota_0 &= \iota(x), \\
[\![f(t_1, \ldots, t_p)]\!]^\iota_{m+d(f)} &= f_{A,k+m}([\![t_1]\!]^\iota_m, \ldots, [\![t_p]\!]^\iota_m), \\
[\![a.f(t_1, \ldots, t_p)]\!]^\iota_{m+d(f)} &= f_{A,k+m}(a, [\![t_1]\!]^\iota_m, \ldots, [\![t_p]\!]^\iota_m), \\
[\![\nu a.f(t_1, \ldots, t_p)]\!]^\iota_{m+d(f)} &= f_{A,k+m}(\langle a \rangle([\![t_1]\!]^\iota_m, \ldots, [\![t_p]\!]^\iota_m)).
\end{aligned}$$

Furthermore, A **satisfies** a Σ-equation $X \vdash_m t = u$ iff for every environment $\iota : X \to A_k$ with $k + m \leq n$, the equality $[\![t]\!]^\iota_m = [\![u]\!]^\iota_m$ holds.

Lemma 3.14. *Let A be a nominal (Σ, n) and $\iota : X \to A_k$ an equivariant environment. Then the evaluation map $[\![\cdot]\!]^\iota_m$ is equivariant.*

Proof. Let $t \in \mathsf{Term}_{\Sigma,m}(X)$ and $\pi \in \mathsf{Perm}(\mathbb{A})$.

We will prove $[\![\pi t]\!]^\iota_m = \pi[\![t]\!]^\iota_m$ by induction on t.

- *For $t = x$ with $x \in X$:*
$$\begin{aligned}
[\![\pi t]\!]^\iota_m &= [\![\pi x]\!]^\iota_0 \\
&= \iota(\pi x) && \text{by Definition 3.13} \\
&= \pi\iota(x) && \text{because } \iota \text{ is equivariant} \\
&= \pi[\![x]\!]^\iota_0 && \text{by Definition 3.13} \\
&= \pi[\![t]\!]^\iota_m.
\end{aligned}$$
- *For $t = f(t_1, \ldots, t_p)$ with $f/p \in \Sigma_0$, $t_1, \ldots, t_p \in \mathsf{Term}_{\Sigma,m'}(X)$, and $m = m' + d(f)$:*
$$\begin{aligned}
[\![\pi t]\!]^\iota_m &= [\![\pi(f(t_1, \ldots, t_p))]\!]^\iota_{m'+d(f)} \\
&= [\![f(\pi t_1, \ldots, \pi t_p)]\!]^\iota_{m'+d(f)} && \text{by Definition 3.5} \\
&= f_{A,m'}([\![\pi t_1]\!]^\iota_{m'}, \ldots, [\![\pi t_p]\!]^\iota_{m'}) && \text{by Definition 3.13} \\
&= f_{A,m'}(\pi[\![t_1]\!]^\iota_{m'}, \ldots, \pi[\![t_p]\!]^\iota_{m'}) && \text{by inductive hypothesis} \\
&= \pi(f_{A,m'}([\![t_1]\!]^\iota_{m'}, \ldots, [\![t_p]\!]^\iota_{m'}) && \text{because } f_{A,m'} \text{ is equivariant} \\
&= \pi[\![f(t_1, \ldots, t_p)]\!]^\iota_{m'+d(f)} && \text{by Definition 3.13} \\
&= \pi[\![t]\!]^\iota_m.
\end{aligned}$$

- *For $t = a.f(t_1, \ldots, t_p)$ and $t = \nu a.f(t_1, \ldots, t_p)$:* Analogous to the above case.

□

Definition 3.15 (Models of Graded Theories)**.** Given a graded theory $T = (\Sigma, E)$ and a depth $n \leq \omega$, a (T, n)**-model** is a nominal (Σ, n)-algebra satisfying every axiom in E.

We define $\mathsf{Alg}(T, n)$ to be the full subcategory of (T, n)-models in $\mathsf{Alg}(\Sigma, n)$.

We will first prove a lemma to describe the behavior of substitutions within an evaluation, which we will then use to prove soundness of the derivation system.

Lemma 3.16 (Substitution Lemma)**.** *Let A be a nominal (Σ, n)-algebra, $\iota : Y \to A_k$ an environment, and $\sigma : X \to \mathsf{Term}_{\Sigma,l}(X)$ a substitution. If $\kappa : X \to A_{k+l}$ with $\kappa(x) = [\![\sigma(x)]\!]^\iota_l$ is equivariant, then $[\![t\sigma]\!]^\iota_{m+l} = [\![t]\!]^\kappa_m$ for every term $t \in \mathsf{Term}_{\Sigma,m}(X)$ with $k + m + l \leq n$.*

Proof. By induction on t.

- *For $t = x$ with $x \in X$:*

$$\begin{aligned} [\![t\sigma]\!]^\iota_{m+l} &= [\![x\sigma]\!]^\iota_l \\ &= [\![\sigma(x)]\!]^\iota_l && \text{by Definition 3.4} \\ &= \kappa(x) && \text{by definition} \\ &= [\![x]\!]^\kappa_0 && \text{by Definition 3.13} \\ &= [\![t]\!]^\kappa_m. \end{aligned}$$

- *For $t = f(t_1, \ldots, t_p)$ with $f/p \in \Sigma_0$, $t_1, \ldots, t_p \in \mathsf{Term}_{\Sigma,m'}(X)$, and $m = m' + d(f)$:*

$$\begin{aligned} [\![t\sigma]\!]^\iota_{m+l} &= [\![(f(t_1, \ldots, t_p))\sigma]\!]^\iota_{m'+d(f)+l} \\ &= [\![f(t_1\sigma, \ldots, t_p\sigma)]\!]^\iota_{m'+d(f)+l} && \text{by Definition 3.4} \\ &= f_{A,m'+l+k}([\![t_1\sigma]\!]^\iota_{m'+l}, \ldots, [\![t_p\sigma]\!]^\iota_{m'+l}) && \text{by Definition 3.13} \\ &= f_{A,m'+l+k}([\![t_1]\!]^\kappa_{m'}, \ldots, [\![t_p]\!]^\kappa_{m'}) && \text{by inductive hypothesis} \\ &= [\![f(t_1, \ldots, t_p)]\!]^\kappa_{m'+d(f)} && \text{by Definition 3.13} \\ &= [\![t]\!]^\kappa_m. \end{aligned}$$

- *For $t = a.f(t_1, \ldots, t_p)$ and $t = \nu a.f(t_1, \ldots, t_p)$:* Analogous to the above case.

□

Theorem 3.17 (Soundness)**.** *The derivation system given in Definition 3.8 is sound: If an equation is derivable in the system, it is also satisfied by every (T, n)-model.*

Proof. Let A be a (T, n)-model and $X \vdash_m t = u$ be derivable in Definition 3.8.

We will show that for every equivariant environment $\iota : X \to A_k$ with $k + m \leq n$, it follows that $[\![t]\!]^\iota_m = [\![u]\!]^\iota_m$, by induction on the derivation of $X \vdash_m t = u$.

- *For (refl):* We know that $t = u = x \in X$, and thus $[\![t]\!]^\iota_m = [\![x]\!]^\iota_m = [\![u]\!]^\iota_m$.
- *For (symm):* We know that $X \vdash_m u = t$ is derivable and thus, by inductive hypothesis, $[\![u]\!]^\iota_m = [\![t]\!]^\iota_m$.
- *For (trans):* We know that, for some v, $X \vdash_m t = v$ and $X \vdash_m v = u$ are derivable. By inductive hypothesis, it follows that $[\![t]\!]^\iota_m = [\![v]\!]^\iota_m = [\![u]\!]^\iota_m$.

- *For (cong$_f$):* We know that $t = f(t_1, \dots, t_p)$, $u = f(u_1, \dots, u_p)$, and $X \vdash_{m'} t_i = u_i$ is derivable for all $i \in \{1, \dots, p\}$ with $m = m' + d(f)$. From this, we can compute

$$\begin{aligned}
[\![t]\!]^\iota_m &= [\![f(t_1, \dots, t_p)]\!]^\iota_{m'+d(f)} \\
&= f_{A,m'}([\![t_1]\!]^\iota_{m'}, \dots, [\![t_p]\!]^\iota_{m'}) && \text{by Definition 3.13} \\
&= f_{A,m'}([\![u_1]\!]^\iota_{m'}, \dots, [\![u_p]\!]^\iota_{m'}) && \text{by inductive hypothesis} \\
&= [\![f(u_1, \dots, u_p)]\!]^\iota_{m'+d(f)} && \text{by Definition 3.13} \\
&= [\![u]\!]^\iota_m.
\end{aligned}$$

- *For (cong$_{a.f}$) and (cong$_{\nu a.f}$):* Analogous to the above case.

- *For (ax$_{r=s}$):* We know that A satisfies $Y \vdash_{m'} r = s$ with $r, s \in \mathsf{Term}_{\Sigma,m'}(Y)$ (because it is a (T, n)-model), and that $t = (\tau r)\sigma, u = (\tau s)\sigma$ for a derivably equivariant substitution $\sigma : Y \to \mathsf{Term}_{\Sigma,l}(X)$ and a permutation $\tau \in \mathsf{Perm}(\mathbb{A})$.

  Let $\kappa : Y \to A_{k+l}$ be defined as $\kappa(x) = [\![\sigma(x)]\!]^\iota_l$. We will first show that κ is equivariant: For $\pi \in \mathsf{Perm}(\mathbb{A})$ and $x \in Y$, we know that $X \vdash_l \pi\sigma(x) = \sigma(\pi x)$ is derivable. It follows that

$$\begin{aligned}
\kappa(\pi x) &= [\![\sigma(\pi x)]\!]^\iota_l && \text{by definition} \\
&= [\![\pi\sigma(x)]\!]^\iota_l && \text{by inductive hypothesis} \\
&= \pi[\![\sigma(x)]\!]^\iota_l && \text{by Lemma 3.14} \\
&= \pi\kappa(x) && \text{by definition.}
\end{aligned}$$

  With this, we can conclude that

$$\begin{aligned}
[\![t]\!]^\iota_m &= [\![(\tau r)\sigma)]\!]^\iota_{m'+l} \\
&= [\![\tau r]\!]^\kappa_{m'} && \text{by Lemma 3.16} \\
&= \tau[\![r]\!]^\kappa_{m'} && \text{by Lemma 3.14} \\
&= \tau[\![s]\!]^\kappa_{m'} && \text{because } A \text{ satisfies } Y \vdash_{m'} r = s \\
&= [\![\tau s]\!]^\kappa_{m'} && \text{by Lemma 3.14} \\
&= [\![(\tau s)\sigma]\!]^\iota_{m'+l} && \text{by Lemma 3.16} \\
&= [\![u]\!]^\iota_m.
\end{aligned}$$

- *For (perm$_f$):* We know that $t = \nu a.f(t_1, \dots, t_p)$ and $u = \nu b.f(u_1, \dots, u_p)$, where $a \# u_i$ and $X \vdash_{m'} t_i = u_i$ is derivable for each $i \in \{1, \dots, p\}$ with $m = m' + d(f)$. Since $a \neq b$ and, by equivariance of $[\![\cdot]\!]^\iota_{m'}$, $a \# [\![u_i]\!]^\iota_{m'}$ for every i, it follows from Proposition 2.8 that

$$\langle a\rangle(a\ b)([\![u_1]\!]^\iota_{m'}, \dots, [\![u_p]\!]^\iota_{m'}) = \langle b\rangle([\![u_1]\!]^\iota_{m'}, \dots, [\![u_p]\!]^\iota_{m'}). \tag{3.1}$$

  It then follows by computation that

$$\begin{aligned}
[\![t]\!]^\iota_m &= [\![\nu a.f(t_1, \dots, t_p)]\!]^\iota_{m'+d(f)} \\
&= f_{A,m'}(\langle a\rangle([\![t_1]\!]^\iota_{m'}, \dots, [\![t_p]\!]^\iota_{m'})) && \text{by Definition 3.13} \\
&= f_{A,m'}(\langle a\rangle([\![(a\ b)u_1]\!]^\iota_{m'}, \dots, [\![(a\ b)u_p]\!]^\iota_{m'})) && \text{by inductive hypothesis} \\
&= f_{A,m'}(\langle a\rangle((a\ b)[\![u_1]\!]^\iota_{m'}, \dots, (a\ b)[\![u_p]\!]^\iota_{m'})) && \text{by Lemma 3.14} \\
&= f_{A,m'}(\langle b\rangle([\![u_1]\!]^\iota_{m'}, \dots, [\![u_p]\!]^\iota_{m'})) && \text{by Equation 3.1} \\
&= [\![\nu b.f(u_1, \dots, u_p)]\!]^\iota_{m'+d(f)} && \text{by Definition 3.13} \\
&= [\![u]\!]^\iota_m.
\end{aligned}$$

□

3.4 Free Models of Graded Theories

In the following section, we will fix a graded theory $T = (\Sigma, E)$, a nominal set X, and a depth $n \le \omega$.

Let $\sim$ be the relation defined as derivable equality in T; i.e., $t \sim u$ for $t, u \in \mathsf{Term}_{\Sigma,m}(X)$ iff $X \vdash_m t = u$ is derivable in T. By Proposition 3.11, $\sim$ is obviously equivariant.

Lemma 3.18. *The relation $\sim$ is a congruence relation.*

Proof. We will show the required properties:

1. $\sim$ is reflexive: We will show that $X \vdash_m t = t$ is derivable for all $t \in \mathsf{Term}_{\Sigma,m}(X)$ by induction on t.
    - *For $t = x$ with $x \in X$:* By applying (refl), we get $X \vdash_0 x = x$.
    - *For $t = f(t_1, \dots, t_p)$ with $f/p \in \Sigma_0$, $t_1, \dots, t_p \in \mathsf{Term}_{\Sigma,m'}(X)$, and $m = m' + d(f)$:* By inductive hypothesis, we can derive $X \vdash_{m'} t_i = t_i$ for every $i \in \{1, \dots, p\}$. By applying (cong$_f$), we then get $X \vdash_m f(t_1, \dots, t_p) = f(t_1, \dots, t_p)$.
    - *For $t = a.f(t_1, \dots, t_p)$ and $t = \nu a.f(t_1, \dots, t_p)$:* Analogous to the above case.
2. $\sim$ is symmetric: Let $t, u \in \mathsf{Term}_{\Sigma,m}(X)$ with $u \sim t$. By definition, we can derive $X \vdash_m u = t$. Applying (symm), we get $X \vdash_m t = u$, and thus $t \sim u$.
3. $\sim$ is transitive: Let $t, u, v \in \mathsf{Term}_{\Sigma,m}(X)$ with $t \sim v$ and $v \sim u$. By definition, we can derive $X \vdash_m t = v$ and $X \vdash_m v = u$. Applying (trans), we get $X \vdash_m t = u$, and thus $t \sim u$.
4. $\sim$ is a congruence: By application of (cong$_f$), (cong$_{a.f}$), and (cong$_{\nu a.f}$) respectively.

□

This allows us to partition $\mathsf{Term}_{\Sigma,m}(X)$ into equivalence classes. We will denote the equivalence class of a term $t \in \mathsf{Term}_{\Sigma,m}(X)$ as $[t]_m \in \mathsf{Term}_{\Sigma,m}(X)/\sim$. The permutation action on the equivalence classes is defined by applying the permutation to the elements of the equivalence class.

Definition 3.19. The (Σ, n)-algebra $F(X)$ is defined as follows:

- $(F(X))_i \coloneqq \mathsf{Term}_{\Sigma,i}(X)/\sim$,
- $f_{F(X),m}([t_1]_m, \dots, [t_p]_m) \coloneqq [f(t_1, \dots, t_p)]_{m+d(f)}$ for $f/p \in \Sigma_0$,
- $f_{F(X),m}(a, [t_1]_m, \dots, [t_p]_m) \coloneqq [a.f(t_1, \dots, t_p)]_{m+d(f)}$ for $f/p \in \Sigma_{\mathsf{f}}$,
- $f_{F(X),m}(\langle a \rangle([t_1]_m, \dots, [t_p]_m)) \coloneqq [\nu a.f(t_1, \dots, t_p)]_{m+d(f)}$ for $f/p \in \Sigma_{\mathsf{b}}$.

Lemma 3.20. *The above description of $F(X)$ is indeed a well-defined nominal (Σ, n)-algebra, in that*

1. *$f_{F(X),m}$ is well-defined and equivariant for all $f \in \Sigma_0$.*
2. *$f_{F(X),m}$ is well-defined and equivariant for all $f \in \Sigma_{\mathsf{f}}$.*
3. *$f_{F(X),m}$ is well-defined and equivariant for all $f \in \Sigma_{\mathsf{b}}$.*

Proof. We will prove each statement individually:

1. To show well-definedness, let $[t_i]_m = [u_i]_m$ for all $i \in \{1, \ldots, p\}$. It follows from (cong$_f$) that $f(t_1, \ldots, t_p) \sim f(u_1, \ldots, u_p)$ and thus

$$\begin{aligned} f_{F(X),m}([t_1]_m, \ldots, [t_p]_m) &= [f(t_1, \ldots, t_p)]_{m+d(f)} \\ &= [f(u_1, \ldots u_p)]_{m+d(f)} \\ &= f_{F(X),m}([u_1]_m, \ldots, [u_p]_m). \end{aligned}$$

   For equivariance, let $\pi \in \mathsf{Perm}(\mathbb{A})$. Then we get

$$\begin{aligned} f_{F(X),m}(\pi[t_1]_m, \ldots, \pi[t_p]_m) &= f_{F(X),m}([\pi t_1]_m, \ldots, [\pi t_p]_m) && \text{by Proposition 2.12} \\ &= [f(\pi t_1, \ldots, \pi t_p)]_{m+d(f)} && \text{by definition} \\ &= [\pi(f(t_1, \ldots, t_p))]_{m+d(f)} && \text{by Definition 3.5} \\ &= \pi[f(t_1, \ldots, t_p)]_{m+d(f)} && \text{by Proposition 2.12} \\ &= \pi(f_{F(X),m}([t_1]_m, \ldots, [t_p]_m)) && \text{by definition.} \end{aligned}$$

2. Analogous to the above case.
3. To show well-definedness, let $\langle a \rangle([t_1]_m, \ldots, [t_p]_m) = \langle b \rangle([u_1]_m, \ldots, [u_p]_m)$. It follows from Proposition 2.8 that we only need to consider two cases:

   If $a = b$, then $[t_i]_m = [u_i]_m$ for every $i \in \{1, \ldots, p\}$ and the statement follows similar to the above case.

   Otherwise, $a \# [u_i]_m$ and $[t_i]_m = (a\ b)[u_i]_m$ for every $i \in \{1, \ldots, p\}$. By Corollary 2.15, we know that there is some $\tilde{u}_i \in \mathsf{Term}_{\Sigma,m}(X)$ with $u_i \sim \tilde{u}_i$ and $a \# \tilde{u}_i$ for every i. It follows from Proposition 2.12 that $[t_i]_m = (a\ b)[\tilde{u}_i]_m = [(a\ b)\tilde{u}_i]_m$ for every i. Thus, we get

$$\begin{aligned} f_{F(X),m}(\langle a \rangle([t_1]_m, \ldots, [t_p]_m)) &= [\nu a.f(t_1, \ldots, t_p)]_{m+d(f)} && \text{by Definition 3.19} \\ &= [\nu b.f(\tilde{u}_1, \ldots, \tilde{u}_p)]_{m+d(f)} && \text{by applying (perm}_f) \\ &= [\nu b.f(u_1, \ldots, u_p)]_{m+d(f)} && \text{by applying (cong}_{\nu b.f}) \\ &= f_{F(X),m}(\langle b \rangle([u_1]_m, \ldots, [u_p]_m)) && \text{by Definition 3.19.} \end{aligned}$$

   Proving equivariance is analogous to the above case.

□

Definition 3.21 (Canonical Environment)**.** The **canonical environment** η_X for $F(X)$ is defined as

$$\begin{aligned} \eta_X &: X \to (F(X))_0, \\ \eta_X(x) &= [x]_0. \end{aligned}$$

Note that, by Proposition 2.12 and Proposition 3.11, η_X is indeed an equivariant environment.

Lemma 3.22. *If* $t \in \mathsf{Term}_{\Sigma,m}(X)$*, then* $[\![t]\!]_m^{\eta_X} = [t]_m$*.*

Proof. By induction on t.

- *For* $t = x$ *with* $x \in X$*:* By definition, we get $[\![t]\!]_m^{\eta_X} = [\![x]\!]_0^{\eta_X} = \eta_X(x) = [x]_0 = [t]_m$.

- *For* $t = f(t_1, \ldots, t_p)$ *with* $f/p \in \Sigma_0$, $t_1, \ldots, t_p \in \mathsf{Term}_{\Sigma,m'}(X)$, *and* $m = m' + d(f)$*:*

$$\begin{aligned}
[\![t]\!]^{\eta_X}_m &= [\![f(t_1, \ldots, t_p)]\!]^{\eta_X}_{m'+d(f)} && \\
&= f_{F(X),m'}([\![t_1]\!]^{\eta_X}_{m'}, \ldots, [\![t_p]\!]^{\eta_X}_{m'}) && \text{by Definition 3.13} \\
&= f_{F(X),m'}([t_1]_{m'}, \ldots, [t_p]^{\eta_X}_{m'}) && \text{by inductive hypothesis} \\
&= [f(t_1, \ldots, t_p)]_{m'+d(f)} && \text{by Definition 3.19} \\
&= [t]_m.
\end{aligned}$$

- *For* $t = a.f(t_1, \ldots, t_p)$ *and* $t = \nu a.f(t_1, \ldots, t_p)$*:* Analogous to the above case.

□

Proposition 3.23. *The above definition of* $F(X)$ *is a* (T, n)*-model.*

Proof. Since we know that $F(X)$ is a nominal (Σ, n)-algebra (by Lemma 3.20), we only need to prove that $F(X)$ satisfies every axiom in E.

Let $Y \vdash_m t = u \in E$ be an axiom and $\iota : Y \to (F(X))_k$ an equivariant environment. We will show that $[\![t]\!]^\iota_m = [\![u]\!]^\iota_m$.

Fix a splitting $u_k : (F(X))_k \to \mathsf{Term}_{\Sigma,k}(X)$ of the canonical projection (in that $[\cdot] \circ u_k = \mathsf{id}$) and let $\sigma = u_k \circ \iota$. This definition of σ yields a derivably equivariant substitution: Let $x \in Y$ and $\pi \in \mathsf{Perm}(\mathbb{A})$. Then

$$\begin{aligned}
[\sigma(\pi x)]_k &= [u_k(\iota(\pi x))]_k && \text{by definition of } \sigma \\
&= \iota(\pi x) && \text{because } [\cdot]_k \circ u_k = \mathsf{id} \\
&= \pi\iota(x) && \text{because } \iota \text{ is equivariant} \\
&= \pi[u_k(\iota(x))]_k && \text{because } [\cdot]_k \circ u_k = \mathsf{id} \\
&= \pi[\sigma(x)]_k && \text{by definition of } \sigma \\
&= [\pi\sigma(x)]_k && \text{by Proposition 2.12,}
\end{aligned}$$

and thus $X \vdash_k \pi\sigma(x) = \sigma(\pi x)$ is derivable. By applying $(\mathrm{ax}_{t=u})$ with $\tau = \mathsf{id}$, we then get $X \vdash_{m+k} t\sigma = u\sigma$.

Since we have

$$\begin{aligned}
\iota(x) &= [u_k(\iota(x))]_k && \text{because } [\cdot]_k \circ u_k = \mathsf{id} \\
&= [\sigma(x)]_k && \text{by definition of } \sigma \\
&= [\![\sigma(x)]\!]^{\eta_X}_k && \text{by Lemma 3.22,}
\end{aligned}$$

we can conclude that

$$\begin{aligned}
[\![t]\!]^\iota_m &= [\![t\sigma]\!]^{\eta_X}_{m+k} && \text{by Lemma 3.16} \\
&= [t\sigma]_{m+k} && \text{by Lemma 3.22} \\
&= [u\sigma]_{m+k} && \text{because } X \vdash_{m+k} t\sigma = u\sigma \\
&= [\![u\sigma]\!]^{\eta_X}_{m+k} && \text{by Lemma 3.22} \\
&= [\![u]\!]^\iota_m && \text{by Lemma 3.16.}
\end{aligned}$$

□

3.5 Inducing a Graded Monad

In this section, we will show that the definition of $F(X)$ from the previous section yields a functor which is a left adjoint to the forgetful functor $\mathsf{Alg}(T, n) \to \mathsf{Nom}$. It will then follow from abstract nonsense that every graded theory over nominal sets induces a graded monad.

We once again fix a graded theory $T = (\Sigma, E)$ and a depth $n \leq \omega$.

Definition 3.24. We define the forgetful functor $G : \mathsf{Alg}(T, n) \to \mathsf{Nom}$ with

$$G(A) = A_0$$
$$G(h) = h_0.$$

Definition 3.25. We define the free functor $F : \mathsf{Nom} \to \mathsf{Alg}(T, n)$ with $F(X)$ as given above and for every $f \in \mathsf{Nom}(X, Y)$

$$(F(f))_i([t]_i) = [t\sigma_f]_i,$$

with the substitution $\sigma_f = (X \xrightarrow{f} Y \hookrightarrow \mathsf{Term}_{\Sigma,0}(Y))$.

Lemma 3.26. *The above description of F is indeed a well-defined functor, in that*

1. *$F(f)$ is a well-defined morphism between (T, n)-algebras for every $f \in \mathsf{Nom}(X, Y)$,*
2. *$F(\mathsf{id}_X) = \mathsf{id}$ for every $X \in \mathsf{Ob}(\mathsf{Nom})$),*
3. *$F(g \circ f) = F(g) \circ F(f)$ for every $f, g \in \mathsf{Nom}(X, Y)$.*

Proof. We will prove the statements individually:

1. Let $f : X \to Y$ be an equivariant function.

   We define an environment $\kappa : X \to (F(Y))_0$ with $\kappa(x) = [\![\sigma_f(x)]\!]_0^{\eta_X}$. Note that κ is indeed equivariant: For every $\pi \in \mathsf{Perm}(\mathbb{A})$, we have $\kappa(\pi x) = [\![f(\pi x)]\!]_0^{\eta_X} = [\![\pi f(x)]\!]_0^{\eta_X} = \pi[\![f(x)]\!]_0^{\eta_X} = \pi\kappa(x)$ by Lemma 3.14 and equivariance of f. It then follows that, for every $[t]_m \in (F(X))_m$, we have

   $$\begin{aligned} (F(f))_m([t]_m) &= [t\sigma_f]_m && \text{by definition} \\ &= [\![t\sigma_f]\!]_m^{\eta_X} && \text{by Lemma 3.22} \\ &= [\![t]\!]_m^{\kappa} && \text{by Lemma 3.16.} \end{aligned}$$

   Thus, we can conclude well-definedness and equivariance from Theorem 3.17 and Lemma 3.14.

   To show Definition 3.12 (1), let $g/p \in \Sigma_0$ and $[t_1]_m, \ldots, [t_p]_m \in (F(X))_m$. Then

   $$\begin{aligned} & (F(f))_{m+d(g)}(g_{F(X),m}([t_1]_m, \ldots, [t_p]_m)) \\ &= (F(f))_{m+d(g)}([g(t_1, \ldots, t_p)]_{m+d(g)}) && \text{by Definition 3.19} \\ &= [(g(t_1, \ldots, t_p))\sigma_f] && \text{by definition} \\ &= [g(t_1\sigma_f, \ldots, t_p\sigma_f)] && \text{by Definition 3.4} \\ &= g_{F(X),m}([t_1\sigma_f]_m, \ldots, [t_p\sigma_f]_m) && \text{by Definition 3.19} \\ &= g_{F(X),m}((F(f))_m([t_1]_m), \ldots, (F(f))_m([t_p]_m)) && \text{by definition.} \end{aligned}$$

   (2) and (3) can be shown with a similar argument.

2. Let $[t] \in (F(X))_m$.

   As seen in (1), we have $(F(\mathsf{id}_X))_i([t]_m) = [\![t]\!]^{\kappa}_m$ for $\kappa : X \to (F(X))$ with $\kappa(x) = [\![\mathsf{id}_X(x)]\!]^{\eta_X}_0 = [\![x]\!]^{\eta_X}_0 = \eta_X(x)$. Thus, $\kappa = \eta_X$ and it follows from Lemma 3.22 that $(F(\mathsf{id}_X))_i([t]_m) = [t]_m = \mathsf{id}([t]_m)$.

3. Let $t \in \mathsf{Term}_{\Sigma,m}(X)$. We will show that $t\sigma_{g\circ f} = (t\sigma_f)\sigma_g$ by induction on t.

   - *For $t = x$ with $x \in X$:* By definition, we get

   $$t\sigma_{g\circ f} = \sigma_{g\circ f}(x) = g(f(x)) = \sigma_g(f(x)) = (f(x))\sigma_g = (\sigma_f(x))\sigma_g = (t\sigma_f)\sigma_g.$$

   - *For $t = f(t_1, \ldots, t_p)$ with $f/p \in \Sigma_0$, $t_1, \ldots, t_p \in \mathsf{Term}_{\Sigma,m'}(X)$, and $m = m' + d(f)$:*

   $$\begin{aligned} t\sigma_{g\circ f} &= f(t_1\sigma_{g\circ f}, \ldots, t_p\sigma_{g\circ f}) && \text{by assumption} \\ &= f((t_1\sigma_f)\sigma_g, \ldots, (t_p\sigma_f)\sigma_g) && \text{by inductive hypothesis} \\ &= (f(t_1\sigma_f, \ldots, t_p\sigma_f))\sigma_g && \text{by Definition 3.4} \\ &= ((f(t_1, \ldots, t_p))\sigma_f)\sigma_g && \text{by Definition 3.4} \\ &= (t\sigma_f)\sigma_g && \text{by assumption.} \end{aligned}$$

   - *For $t = a.f(t_1, \ldots, t_p)$ and $t = \nu a.f(t_1, \ldots, t_p)$:* Analogous to the above case.

   With this, we can conclude that

   $$\begin{aligned} (F(g\circ f))_m([t]_m) &= [t\sigma_{g\circ f}]_m \\ &= [(t\sigma_f)\sigma_g]_m \\ &= (F(g))_m([t\sigma_f]_m) \\ &= (F(g))_m((F(f))_m([t]_m) \\ &= (F(g)\circ F(f))_m([t]_m). \end{aligned}$$

□

Theorem 3.27. *The functor F is a left-adjoint to the forgetful functor G with the unit η as defined in Definition 3.21 and the counit ε given by*

$$\begin{aligned} \varepsilon_A &: F(A_0) \to A, \\ (\varepsilon_A)_i([t]_i) &= [\![t]\!]^{\mathsf{id}}_i. \end{aligned}$$

Proof. We will first show that ε_A is indeed a well-defined morphism between algebras for every $A \in \mathsf{Ob}(\mathsf{Alg}(T, n))$:

- Well-definedness and equivariance follow directly from Theorem 3.17 and Lemma 3.14.
- To show Definition 3.12 (1), let $f/p \in \Sigma_0$ and $[t_1]_m, \ldots, [t_p]_m \in (F(A_0))_m$. Then

$$\begin{aligned} &(\varepsilon_A)_{m+d(f)}(f_{F(A_0),m}([t_1]_m, \ldots, [t_p]_m)) \\ &= (\varepsilon_A)_{m+d(f)}([f(t_1, \ldots, t_p)]_{m+d(f)}) && \text{by Definition 3.19} \\ &= [\![f(t_1, \ldots, t_p)]\!]^{\mathsf{id}}_{m+d(f)} && \text{by definition} \\ &= f_{A,m}([\![t_1]\!]^{\mathsf{id}}_m, \ldots, [\![t_p]\!]^{\mathsf{id}}_m) && \text{by Definition 3.13} \\ &= f_{A,m}((\varepsilon_A)_m([t_1]_m), \ldots, (\varepsilon_A)_m([t_p]_m)) && \text{by definition.} \end{aligned}$$

- With a similar argument, one can also show (2) and (3).

Next, we will show that η is indeed natural. So let $X, Y \in \mathsf{Ob}(\mathsf{Nom})$, $f \in \mathsf{Nom}(X, Y)$, and $x \in X$. Let σ_f be defined as in Definition 3.25. Then

$$\begin{aligned}
\eta_Y(f(x)) &= [f(x)]_m && \text{by Definition 3.21} \\
&= [\sigma_f(x)]_m && \text{by definition of } \sigma_f \\
&= [x\sigma_f]_m && \text{by Definition 3.4} \\
&= (F(f))_m([x]) && \text{by Definition 3.25} \\
&= (F(f))_m(\eta_X(x)) && \text{by Definition 3.21.}
\end{aligned}$$

To show that ε is also natural, let $A, B \in \mathsf{Ob}(\mathsf{Alg}(T, n))$ and $h \in \mathsf{Alg}(T, n)(A, B)$. We will show $(\varepsilon_B)_m((F(h_0))_m([t]_m)) = h_m((\varepsilon_A)_m([t]_m))$ for every $t \in \mathsf{Term}_{\Sigma,m}(A_0)$ by induction on t.

- *For $t = x$ with $x \in A_0$:*

$$\begin{aligned}
(\varepsilon_B)_0((F(h_0))_0([x]_0)) &= (\varepsilon_B)_0([x\sigma_{h_0}]_0) && \text{by Definition 3.25} \\
&= (\varepsilon_B)_0([h_0(x)]_0) && \text{by Definition 3.4} \\
&= [\![h_0(x)]\!]_0^{\mathsf{id}} && \text{by definition of } \varepsilon \\
&= h_0(x) && \text{by Definition 3.13} \\
&= h_0([\![x]\!]_0^{\mathsf{id}}) && \text{by Definition 3.13} \\
&= h_0((\varepsilon_A)_0([x])) && \text{by definition of } \varepsilon.
\end{aligned}$$

- *For $t = f(t_1, \ldots, t_p)$ with $f/p \in \Sigma_0$, $t_1, \ldots, t_p \in \mathsf{Term}_{\Sigma,m'}(X)$, and $m = m' + d(f)$:*

$$\begin{aligned}
&(\varepsilon_B)_m((F(h_0))_m([f(t_1, \ldots, t_p)]_m)) \\
&= (\varepsilon_B)_m((F(h_0))_m(f_{F(A_0),m'}([t_1]_{m'}, \ldots, [t_p]_{m'}))) && \text{by Definition 3.19} \\
&= (\varepsilon_B)_m(f_{F(B_0),m'}((F(h_0))_{m'}([t_1]_{m'}), \ldots, (F(h_0))_{m'}([t_p]_{m'}))) && \text{by Definition 3.12 (1)} \\
&= f_{B,m'}((\varepsilon_B)_{m'}((F(h_0))_{m'}([t_1]_{m'})), \ldots, (\varepsilon_B)_{m'}((F(h_0))_{m'}([t_p]_{m'}))) && \text{by Definition 3.12 (1)} \\
&= f_{B,m'}(h_{m'}((\varepsilon_A)_{m'}([t_1]_{m'})), \ldots, h_{m'}((\varepsilon_A)_{m'}([t_p]_{m'}))) && \text{inductive hypothesis} \\
&= h_m(f_{A,m'}((\varepsilon_A)_{m'}([t_1]_{m'}), \ldots, (\varepsilon_A)_{m'}([t_p]_{m'}))) && \text{by Definition 3.12 (1)} \\
&= h_m((\varepsilon_A)_m(f_{F(A_0),m'}([t_1]_{m'}, \ldots, [t_p]_{m'}))) && \text{by Definition 3.12 (1)} \\
&= h_m((\varepsilon_A)_m([f(t_1, \ldots, t_p)]_m)) && \text{by Definition 3.19.}
\end{aligned}$$

- *For $t = a.f(t_1, \ldots, t_p)$ and $t = \nu a.f(t_1, \ldots, t_p)$:* Analogous to the above case.

Finally, we will prove the adjunction using the counit-unit equations

$$\mathsf{Id}_F = \varepsilon F \circ F\eta, \tag{3.2}$$

$$\mathsf{Id}_G = G\varepsilon \circ \eta G. \tag{3.3}$$

Let $A \in \mathsf{Ob}(\mathsf{Alg}(T, n))$ and $X \in \mathsf{Ob}(\mathsf{Nom})$.

For Equation 3.2, we will show $(\varepsilon_{F(X)})_m((F(\eta_X))_m([t]_m)) = [t]_m$ for every $t \in \mathsf{Term}_{\Sigma,m}(X)$ by induction on t.

- *For $t = x$ with $x \in X$:*

$$\begin{aligned}
(\varepsilon_{F(X)})_0((F(\eta_X))_0([x]_0)) &= (\varepsilon_{F(X)})_0([x\sigma_{\eta_X}]_0) && \text{by Definition 3.25} \\
&= (\varepsilon_{F(X)})_0([[x]_0]_0) && \text{by Definition 3.4} \\
&= [\![[x]_0]\!]_0^{\mathsf{id}} && \text{by definition of } \varepsilon \\
&= [x]_0 && \text{by Definition 3.13.}
\end{aligned}$$

- *For $t = f(t_1, \dots, t_p)$ with $f/p \in \Sigma_0$, $t_1, \dots, t_p \in \mathsf{Term}_{\Sigma,m'}(X)$, and $m = m' + d(f)$:*

$$
\begin{aligned}
& (\varepsilon_{F(X)})_m((F(\eta_X))_m([f(t_1, \dots, t_p)]_m)) \\
= \quad & \text{(by Definition 3.19)} \\
& (\varepsilon_{F(X)})_m((F(\eta_X))_m(f_{F(X),m'}([t_1]_{m'}, \dots, [t_p]_{m'}))) \\
= \quad & \text{(by Definition 3.12 (1))} \\
& (\varepsilon_{F(X)})_m(f_{F((F(X))_0),m'}((F(\eta_X))_{m'}([t_1]_{m'}), \dots, (F(\eta_X))_{m'}([t_p]_{m'}))) \\
= \quad & \text{(by Definition 3.12 (1))} \\
& f_{F(X),m'}((\varepsilon_{F(X)})_{m'}((F(\eta_X))_{m'}([t_1]_{m'})), \dots, (\varepsilon_{F(X)})_{m'}((F(\eta_X))_{m'}([t_p]_{m'}))) \\
= \quad & \text{(by inductive hypothesis)} \\
& f_{F(X),m'}([t_1]_{m'}, \dots [t_p]_{m'}) \\
= \quad & \text{(by Definition 3.19)} \\
& [f(t_1, \dots, t_p)]_m.
\end{aligned}
$$

- *For $t = a.f(t_1, \dots, t_p)$ and $t = \nu a.f(t_1, \dots, t_p)$:* Analogous to the above case.

For Equation 3.3, let $x \in A_0$. Then we have

$$
\begin{aligned}
(\varepsilon_A)_0(\eta_{A_0}(x)) &= (\varepsilon_A)_0([x]_0) && \text{by Definition 3.21} \\
&= [\![x]\!]_0^{\mathsf{id}} && \text{by definition of } \varepsilon \\
&= x && \text{by Definition 3.13.}
\end{aligned}
$$

With this, it follows that $(\varepsilon, \eta) : F \dashv G$. □

This adjunction can be used to show that every graded theory induces the following graded monad over Nom:

Definition 3.28. Given a graded theory $T = (\Sigma, E)$, the graded monad $((M_n)_{n \in \mathbb{N}_0}, \eta, (\mu^{nk})_{n,k \in \mathbb{N}_0})$ over Nom induced by it is defined as

$$
\begin{gathered}
M_n : \mathsf{Nom} \to \mathsf{Nom}, \\
M_n(X) = (F(X))_n = \mathsf{Term}_{\Sigma,n}(X)/\sim, \\
M_n(f)([t]_n) = (F(f))_n = [t\sigma_f]_n, \\
\\
\eta_X : X \to M_0(X), \\
\eta_X(x) = [x]_0, \\
\\
\mu_X^{nk} : M_n(M_k(X)) \to M_{n+k}(X), \\
\mu_X^{nk}([t]_n) = [\![t]\!]_n^{\mathsf{id}},
\end{gathered}
$$

where σ_f is defined as in Definition 3.25.

Corollary 3.29. *The above definition is indeed a graded monad.*

Proof. We set $n = \omega$ and use the adjunction $F \dashv G$ provided by Theorem 3.27.

Let $\star : \mathbb{N}_0 \times \mathsf{Alg}(T,\omega) \to \mathsf{Alg}(T,\omega)$ be the strict action of the discrete monoidal category $(\mathbb{N}_0, +, 0)$ on $\mathsf{Alg}(T,\omega)$ defined as

$$n \star ((A_i), (f_{A,i})) = ((A_{i+n}), (f_{A,i+n})),$$
$$n \star (f_i) = (f_{i+n}).$$

We can now conclude [FKM16, Section 3] that there is a graded monad $((M_n), \eta, (\mu^{nk}))$ with

$$M_n(X) = G(n \star F(X)) = (n \star F(X))_0 = (F(X))_n = \mathsf{Term}_{\Sigma,n}(X)/{\sim},$$
$$M_n(f)([t]_n) = G(n \star F(f))([t]_n) = (n \star F(f))_0([t]_n) = (F(f))_n([t]_n) = [t\sigma_f]_n,$$

η being the unit of the adjunction and

$$\mu^{nk}_X : M_n(M_k(X)) = G(n \star F(G(k \star F(X)))) \to G(n \star (k \star F(X))) = M_{n+k}(X),$$
$$\mu^{nk}_X([t]_n) = G(n \star \varepsilon_{k \star F(X)})([t]_n) = (\varepsilon_{k \star F(X)})_n([t]_n) = [\![t]\!]^{\mathsf{id}}_n.$$

□

Intuitively, μ^{nk} "collapses" the inner equivalence classes in a term:

Lemma 3.30. *Let $T = (\Sigma, E)$ be a graded theory and $((M_n), \eta, (\mu^{nk}))$ be the graded monad induced by it. If $\sigma : M_k(X) \to \mathsf{Term}_{\Sigma,k}(X)$ is a splitting (in that $[\cdot] \circ \sigma = \mathsf{id}$) and $t \in \mathsf{Term}_{\Sigma,n}(M_k(X))$, then $\mu^{nk}_X([t]_n) = [t\sigma]_{n+k}$.*

Proof. First note that, for $[u]_k \in M_k(X)$, we have

$$\begin{aligned}[\![\sigma([u]_k)]\!]^{\eta_X}_k &= [\sigma([u]_k)]_k && \text{by Lemma 3.22}\\ &= \mathsf{id}([u]_k) && \text{because } [\cdot] \circ \sigma = \mathsf{id}.\end{aligned}$$

It then follows that, for $t \in \mathsf{Term}_{\Sigma,n}(M_k(X))$,

$$\begin{aligned}\mu^{nk}_X([t]_n) &= [\![t]\!]^{\mathsf{id}}_n && \text{by Definition 3.28}\\ &= [\![t\sigma]\!]^{\eta_X}_{n+k} && \text{by Lemma 3.16 with } \kappa = \mathsf{id}\\ &= [t\sigma]_{n+k} && \text{by Lemma 3.22.}\end{aligned}$$

□

We can show that substitutions can be expressed as first substituting variables with variables of equivalence classes of terms followed by a multiplication:

Lemma 3.31. *Let $T = (\Sigma, E)$ be a graded theory and $((M_n), \eta, (\mu^{nk}))$ be the graded monad induced by it. If $t \in \mathsf{Term}_{\Sigma,n}(Y)$ and $\sigma : Y \to \mathsf{Term}_{\Sigma,k}(X)$, then $[t\sigma]_{n+k} = \mu^{n,k}_X([t\bar{\sigma}]_n)$, where $\bar{\sigma} : Y \to \mathsf{Term}_{\Sigma,0}(M_k(X))$ is defined as $\bar{\sigma}(x) = [\sigma(x)]_k$.*

Proof. By induction on t.

- *For $t = x$ with $x \in Y$:*

$$[t\sigma]_{n+k} = [\sigma(x)]_k = \bar{\sigma}(x) = [\![\bar{\sigma}(x)]\!]^{\mathsf{id}}_0 = \mu^{0,k}_X([\bar{\sigma}(x)]_0) = \mu^{n,k}_X([t\bar{\sigma}]_n).$$

- *For $t = f(t_1, \dots, t_p)$ with $f/p \in \Sigma_0$, $t_1, \dots, t_p \in \mathsf{Term}_{\Sigma,m}(Y)$, and $1 = m + d(f)$:* By inductive hypothesis, we know that $[t_i\sigma]_{m+k} = \mu_X^{m,k}([t_i\bar{\sigma}]_m)$ for every i. It then follows that

$$\begin{aligned}
[t\sigma]_{n+k} &= [f(t_1\sigma, \dots, t_p\sigma)]_{n+k} && \text{by Definition 3.4} \\
&= f_{F(X),m+k}([t_1\sigma]_{m+k}, \dots, [t_p\sigma]_{m+k}) && \text{by Definition 3.19} \\
&= f_{F(X),m+k}(\mu_X^{m,k}([t_1\bar{\sigma}]_m), \dots, \mu_X^{m,k}([t_p\bar{\sigma}]_m)) && \text{by inductive hypothesis} \\
&= f_{F(X),m+k}([\![t_1\bar{\sigma}]\!]_m^{\mathsf{id}}, \dots, [\![t_p\bar{\sigma}]\!]_m^{\mathsf{id}}) && \text{by Definition 3.28} \\
&= [\![f(t_1\bar{\sigma}, \dots, t_p\bar{\sigma})]\!]_n^{\mathsf{id}} && \text{by Definition 3.13} \\
&= \mu_X^{n,k}([t\bar{\sigma}]_n) && \text{by Definition 3.28.}
\end{aligned}$$

- *For $t = a.f(t_1, \dots, t_p)$ and $t = \nu a.f(t_1, \dots, t_p)$:* Analogous to the above case.

□

We will also show the following lemma for operations:

Lemma 3.32. *Let $T = (\Sigma, E)$ be a graded theory and $((M_n), \eta, (\mu^{nk}))$ be the graded monad induced by it. For $t_1, \dots, t_p \in \mathsf{Term}_{\Sigma,n}(X)$:*

- *If $f/p \in \Sigma_0$, then $[f(t_1, \dots, t_p)]_{n+d(f)} = \mu_X^{d(f),n}([f([t_1]_n, \dots, [t_p]_n)]_{d(f)}$.*
- *If $f/p \in \Sigma_{\mathsf{f}}$, then $[a.f(t_1, \dots, t_p)]_{n+d(f)} = \mu_X^{d(f),n}([a.f([t_1]_n, \dots, [t_p]_n)]_{d(f)}$.*
- *If $f/p \in \Sigma_{\mathsf{b}}$, then $[\nu a.f(t_1, \dots, t_p)]_{n+d(f)} = \mu_X^{d(f),n}([\nu a.f([t_1]_n, \dots, [t_p]_n)]_{d(f)}$.*

Proof. We will show the first statement by simple computation:

$$\begin{aligned}
\mu_X^{d(f),n}([f([t_1]_n, \dots, [t_p]_n)]_{d(f)} &= [\![f([t_1]_n, \dots, [t_p]_n)]\!]_{d(f)}^{\mathsf{id}} && \text{by Definition 3.28} \\
&= f_{F(X),n}([\![[t_1]_n]\!]_0^{\mathsf{id}}, \dots, [\![[t_p]_n]\!]_0^{\mathsf{id}}) && \text{by Definition 3.13} \\
&= f_{F(X),n}([t_1]_n, \dots, [t_p]_n) && \text{by Definition 3.13} \\
&= [f(t_1, \dots, t_p)]_{n+d(f)} && \text{by Definition 3.19.}
\end{aligned}$$

□

3.6 Depth-1 Graded Monads

We will first define a notion of depth-1 graded theories over Nom.

Definition 3.33 (Depth-1 Graded Theories)**.** A graded theory $T = (\Sigma, E)$ is **depth-1** if all its operations and axioms are at most depth-1.

We can then show that every such graded theory induces a graded monad that is depth-1 according to Definition 2.27.

Fix a depth-1 graded theory $T = (\Sigma, E)$ and let $((M_n), \eta, (\mu^{nk}))$ be the graded monad induced by it as described in Definition 3.28.

The monad multiplication $\mu_X^{1,n}$ collapses depth-1 terms with embedded depth-n terms into terms of depth $1 + n$. Conversely, we can split a depth-$1 + n$ term into such a layered term, given that all operations are at most depth-1.

To do this, let $s^n_X : \mathsf{Term}_{\Sigma,1+n}(X) \to \mathsf{Term}_{\Sigma,1}(M_n(X))$ be defined as

$$\begin{aligned} s^n_X(f(t_1,\ldots,t_p)) &= f(s^n_X(t_1),\ldots,s^n_X(t_p)) && \text{if } f/p \in \Sigma_0 \text{ and } d(f)=0, \\ s^n_X(f(t_1,\ldots,t_p)) &= f([t_1]_n,\ldots,[t_p]_n) && \text{if } f/p \in \Sigma_0 \text{ and } d(f)=1, \end{aligned}$$

and similarly for Σ_{f} and Σ_{b}.

Lemma 3.34. *The function s^n_X is equivariant.*

Proof. Let $\pi \in \mathsf{Perm}(\mathbb{A})$. We will show $s^n_X(\pi t) = \pi s^n_X(t)$ for all $t \in \mathsf{Term}_{\Sigma,1+n}$ by induction on t, only considering cases where t has at least uniform depth 1.

- *For $t = f(t_1,\ldots,t_p)$ with $f/p \in \Sigma_0$, $t_1,\ldots,t_p \in \mathsf{Term}_{\Sigma,m}(X)$, and $1+n = m+d(f)$:* Since all operations are at most depth-1, we only have to consider two cases:

  If $d(f)=0$, then

$$\begin{aligned} s^n_X(\pi t) &= s^n_X(f(\pi t_1,\ldots,\pi t_p)) && \text{by Definition 3.5} \\ &= f(s^n_X(\pi t_1),\ldots,s^n_X(\pi t_p)) && \text{by definition of } s^n_X \\ &= f(\pi s^n_X(t_1),\ldots,\pi s^n_X(t_p)) && \text{by inductive hypothesis} \\ &= \pi f(s^n_X(t_1),\ldots,s^n_X(t_p)) && \text{by Definition 3.5} \\ &= \pi s^n_X(t) && \text{by definition of } s^n_X. \end{aligned}$$

  If $d(f)=1$, then

$$\begin{aligned} s^n_X(\pi t) &= s^n_X(f(\pi t_1,\ldots,\pi t_p)) && \text{by Definition 3.5} \\ &= f([\pi t_1]_n,\ldots,[\pi t_p]_n) && \text{by definition of } s^n_X \\ &= f(\pi[t_1]_n,\ldots,\pi[t_p]_n) && \text{by Proposition 2.12} \\ &= \pi f([t_1]_n,\ldots,[t_p]_n) && \text{by Definition 3.5} \\ &= \pi s^n_X(t) && \text{by definition of } s^n_X. \end{aligned}$$

- *For $t = a.f(t_1,\ldots,t_p)$ and $t = \nu a.f(t_1,\ldots,t_p)$:* Analogous to the above case.

□

Of course, where we split exactly is chosen arbitrarily: Here, we use the outermost term inside a depth-1 operation, however this term could have more depth-0 operations we could include. We will see that this doesn't matter in the context of morphisms satisfying the universal property of a coequalizer. To do this, we will first show some general properties of such morphisms:

Lemma 3.35. *Let (Q,h) be an M_0-algebra and $q : M_1(M_n(X)) \to Q$ a morphism between M_0-algebras such that $q \circ M_1(\mu^{0,n}_X) = q \circ \mu^{1,0}_{M_n(X)}$.*

1. *If $t_1,\ldots,t_p \in \mathsf{Term}_{\Sigma,1}(M_n(X))$, then*
   - $q([f(t_1,\ldots,t_p)]_1) = h([f(q([t_1]_1),\ldots,q([t_p]_1))]_0)$ *for $f/p \in \Sigma_0$, $d(f)=0$,*
   - $q([a.f(t_1,\ldots,t_p)]_1) = h([a.f(q([t_1]_1),\ldots,q([t_p]_1))]_0)$ *for $f/p \in \Sigma_{\mathsf{f}}$, $d(f)=0$,*
   - $q([\nu a.f(t_1,\ldots,t_p)]_1) = h([\nu a.f(q([t_1]_1),\ldots,q([t_p]_1))]_0)$ *for $f/p \in \Sigma_{\mathsf{b}}$, $d(f)=0$.*
2. *If $t_1,\ldots,t_p \in \mathsf{Term}_{\Sigma,0}(M_n(X))$, then*
   - $q([f(t_1,\ldots,t_p)]_1) = q([f(\mu^{0,n}_X([t_1]_0),\ldots,\mu^{0,n}_X([t_p]_0))]_1)$ *for $f/p \in \Sigma_0$, $d(f)=1$,*

- $q([a.f(t_1,\dots,t_p)]_1) = q([a.f(\mu_X^{0,n}([t_1]_0),\dots,\mu_X^{0,n}([t_p]_0))]_1)$ *for* $f/p \in \Sigma_{\mathsf{f}}$, $d(f)=1$,
- $q([\nu a.f(t_1,\dots,t_p)]_1) = q([\nu a.f(\mu_X^{0,n}([t_1]_0),\dots,\mu_X^{0,n}([t_p]_0))]_1)$ *for* $f/p \in \Sigma_{\mathsf{b}}$, $d(f)=1$.

3. *If* $t \in \mathsf{Term}_{\Sigma,0}(Y)$ *and* $\sigma : Y \to \mathsf{Term}_{\Sigma,1+n}(X)$, *then* $q([s_X^n(t\sigma)]_1) = h([t\bar{\sigma}]_0)$ *for* $\bar{\sigma} : Y \to \mathsf{Term}_{\Sigma,0}(Q)$ *with* $\bar{\sigma}(x) = q([s_X^n(\sigma(x))]_1)$.
4. *If* $t \in \mathsf{Term}_{\Sigma,1}(Y)$ *and* $\sigma : Y \to \mathsf{Term}_{\Sigma,n}(X)$, *then* $q([s_X^n(t\sigma)]_1) = q([t\bar{\sigma}]_1)$ *for* $\bar{\sigma} : Y \to \mathsf{Term}_{\Sigma,0}(M_n(X))$ *with* $\bar{\sigma}(x) = [\sigma(x)]_n$.

Proof. Since q is a morphism between the M_0-algebras $(M_1(M_n(X)), \mu^{0,1}_{M_n(X)})$ and (Q,h), we know that

$$h \circ M_0(q) = q \circ \mu^{0,1}_{M_n(X)}. \tag{3.4}$$

We will proceed to show each statement individually:

1. We will only show the statement for Σ_0, the arguments for the others are completely analogous.

   By computation, we get

$$\begin{aligned} q([f(t_1,\dots,t_p)]_1) &= q(\mu^{0,1}_{M_n(X)}([f([t_1]_1,\dots,[t_p]_1)]_0)) && \text{by Lemma 3.32}\\ &= h(M_0(q)([f([t_1]_1,\dots,[t_p]_1)]_0)) && \text{by Equation 3.4}\\ &= h([f([t_1]_1,\dots,[t_p]_1)\sigma_q]_0) && \text{by Definition 3.28}\\ &= h([f(q([t_1]_1),\dots,q([t_p]_1))]_0) && \text{by Definition 3.4.} \end{aligned}$$

2. Once again, we will only show the statement for Σ_0.

   By computation, we get

$$\begin{aligned} q([f(t_1,\dots,t_p)]_1) &= q(\mu^{1,0}_{M_n(X)}([f([t_1]_0,\dots,[t_p]_0)]_1)) && \text{by Lemma 3.32}\\ &= q(M_1(\mu_X^{0,n})([f([t_1]_0,\dots,[t_p]_0)]_1)) && \text{by assumption}\\ &= q([f([t_1]_0,\dots,[t_p]_0)\sigma_{\mu_X^{0,n}}]_1 && \text{by Definition 3.28}\\ &= q([f(\mu_X^{0,n}([t_1]_0),\dots,\mu_X^{0,n}([t_p]_0))]_1) && \text{by Definition 3.4.} \end{aligned}$$

3. By induction on t.

   - *For* $t = x$ *with* $x \in Y$*:*

$$\begin{aligned} q([s_X^n(t\sigma)]_1) &= q(\llbracket [s_X^n(t\sigma)]_1 \rrbracket_0^{\mathsf{id}}) && \text{by Definition 3.13}\\ &= q(\mu^{0,1}_{M_n(X)}([[s_X^n(t\sigma)]_1]_0)) && \text{by Definition 3.28}\\ &= h(M_0(q)([[s_X^n(t\sigma)]_1]_0)) && \text{by Equation 3.4}\\ &= h([q([s_X^n(\sigma(x))]_1)]_0) && \text{by Definition 3.28}\\ &= h([\bar{\sigma}(x)]_0) && \text{by definition of } \bar{\sigma}\\ &= h([t\bar{\sigma}]_0) && \text{by Definition 3.4.} \end{aligned}$$

   - *For* $t = f(t_1,\dots,t_p)$ *with* $f/p \in \Sigma_0$*:* We know that $d(f) = 0$ (because t is depth-0).

It then follows that

$$
\begin{aligned}
& q([s_X^n(t\sigma)]_1) \\
&= q([s_X^n(f(t_1\sigma,\dots,t_p\sigma))]_1) && \text{by Definition 3.4} \\
&= q([f(s_X^n(t_1\sigma),\dots,s_X^n(t_p\sigma))]_1) && \text{by definition of } s_X^n \\
&= h([f(q([s_X^n(t_1\sigma)]_1),\dots,q([s_X^n(t_p\sigma)]_1)]_0) && \text{by (1)} \\
&= h([f(h([t_1\bar{\sigma}]_0),\dots,h([t_p\bar{\sigma}]_0))]_0) && \text{by inductive hypothesis} \\
&= h(M_0(h)([f([t_1\bar{\sigma}]_0,\dots,[t_p\bar{\sigma}]_0)]_0) && \text{by Definition 3.28} \\
&= h(\mu_Q^{0,0}([f([t_1\bar{\sigma}]_0,\dots,[t_p\bar{\sigma}]_0)]_0)) && \text{because } (Q,h) \text{ is an } M_0\text{-algebra} \\
&= h([f(t_1\bar{\sigma},\dots,t_p\bar{\sigma})]_0) && \text{by Lemma 3.32} \\
&= h([t\bar{\sigma}]_0) && \text{by Definition 3.4.}
\end{aligned}
$$

- *For $t = a.f(t_1,\dots,t_p)$ and $t = \nu a.f(t_1,\dots,t_p)$:* Analogous to the above case.

4. By induction on t.
    - *For $t = x$ with $x \in Y$:* This would imply that t is depth-0, but we know that t has uniform depth 1.
    - *For $t = f(t_1,\dots,t_p)$ with $f/p \in \Sigma_0$:* Since t has uniform depth 1, we have to consider two cases:

      For $d(f) = 0$, we get

$$
\begin{aligned}
q([s_X^n(t\sigma)]_1) &= q([s_X^n(f(t_1\sigma,\dots,t_p\sigma))]_1) && \text{by Definition 3.4} \\
&= q([f(s_X^n(t_1\sigma),\dots,s_X^n(t_p\sigma))]_1) && \text{by definition of } s_X^n \\
&= h([f(q([s_X^n(t_1\sigma)]_1),\dots,q([s_X^n(t_p\sigma)]_1)]_0) && \text{by (1)} \\
&= h([f(q([t_1\bar{\sigma}]_1),\dots,q([t_p\bar{\sigma}]_1)]_0) && \text{by inductive hypothesis} \\
&= q([f(t_1\bar{\sigma},\dots,t_p\bar{\sigma})]_1) && \text{by (1)} \\
&= q([t\bar{\sigma}]_1) && \text{by Definition 3.4.}
\end{aligned}
$$

      For $d(f) = 1$, we get

$$
\begin{aligned}
q([s_X^n(t\sigma)]_1) &= q([s_X^n(f(t_1\sigma,\dots,t_p\sigma))]_1) && \text{by Definition 3.4} \\
&= q([f([t_1\sigma]_n,\dots,[t_p\sigma]_n)]_1) && \text{by definition of } s_X^n \\
&= q([f(\mu_X^{0,n}([t_1\bar{\sigma}]_0),\dots,\mu_X^{0,n}([t_p\bar{\sigma}]_0)]))]_1) && \text{by Lemma 3.31} \\
&= q([f(t_1\bar{\sigma},\dots,t_p\bar{\sigma})]_1) && \text{by (2)} \\
&= q([t\bar{\sigma}]_1) && \text{by Definition 3.4.}
\end{aligned}
$$

    - *For $t = a.f(t_1,\dots,t_p)$ and $t = \nu a.f(t_1,\dots,t_p)$:* Analogous to the above case.

□

With this, we can finally show the coequalizer property for $\mu_X^{1,n}$ and thus the desired result.

Theorem 3.36. *The graded monad induced by a depth-1 graded theory $T = (\Sigma, E)$ is depth-1.*

Proof. We will show that $\mu_X^{1,n}$ is a coequalizer in the category of M_0-algebras for $M_1(\mu_X^{0,n})$ and $\mu_{M_n(X)}^{1,0}$ as given in Definition 2.27 for all $X \in \mathsf{Ob}(\mathsf{Nom})$ and $n \in \mathbb{N}_0$.

By Definition 2.23, we already know that $\mu_X^{1,n} \circ M_1(\mu_X^{0,n}) = \mu_X^{1,n} \circ \mu_{M_n(X)}^{1,0}$.

Now let (Q, h) be an M_0-algebra and $q : M_1(M_n(X)) \to Q$ a morphism between M_0-algebras such that $q \circ M_1(\mu_X^{0,n}) = q \circ \mu_{M_n(X)}^{1,0}$. We now have to show that there exists a unique morphism $\bar{q} : M_{1+n}(X) \to Q$ with $\bar{q} \circ \mu_X^{1,n} = q$.

So let $\bar{q}$ be defined as $\bar{q}([t]_{1+n}) = q([s_X^n(t)]_1)$. We will show that this definition satisfies our requirements:

1. If $\pi \in \mathsf{Perm}(\mathbb{A})$ and $t \in \mathsf{Term}_{\Sigma,1+n}(X)$, then

$$\begin{aligned}
\bar{q}([\pi t]_{1+n}) &= q([s_X^n(\pi t)]_1) && \text{by definition of } \bar{q} \\
&= q([\pi s_X^n(t)]_1) && \text{by Lemma 3.34} \\
&= q(\pi[s_X^n(t)]_1) && \text{by Proposition 2.12 and Proposition 3.11} \\
&= \pi q([s_X^n(t)]_1) && \text{by equivariance of } q \\
&= \pi \bar{q}([t]_{1+n}) && \text{by definition of } \bar{q}.
\end{aligned}$$

   Furthermore, it follows by equivariance of s_X^n, the canonical projection, and q, that

$$\mathsf{supp}(\bar{q}([t]_{1+n})) = \mathsf{supp}(q([s_X^n(t)]_1)) \subseteq \mathsf{supp}(t).$$

2. $\bar{q}$ is well-defined; i.e., if $X \vdash_{1+n} t = u$ is derivable, then $\bar{q}([t]_{1+n}) = \bar{q}([u]_{1+n})$. We will show this by induction on the derivation.

   - *For (refl):* This would imply that $t = x$ for some $x \in X$, however t has uniform depth $1 + n$, whereas x has uniform depth 0.
   - *For (symm):* We know that $X \vdash_{1+n} u = t$ is derivable and, by inductive hypothesis, we get $\bar{q}([u]_{1+n}) = \bar{q}([t]_{1+n})$.
   - *For (trans):* We know that $X \vdash_{1+n} t = v$ and $X \vdash_{1+n} v = u$ are derivable for some v. Thus, by inductive hypothesis, we get $\bar{q}([t]_{1+n}) = \bar{q}([v]_{1+n}) = \bar{q}([u]_{1+n})$.
   - *For (cong$_f$):* We know that $t = f(t_1, \ldots, t_p)$, $u = f(u_1, \ldots, u_p)$, and $X \vdash_m t_i = u_i$ is derivable for all $i \in \{1, \ldots, p\}$ with $1 + n = m + d(f)$. Since the operations are at most depth-1, we only have to consider two cases:

     If $d(f) = 0$, we get

$$\begin{aligned}
\bar{q}([t]_{1+n}) &= q([s_X^n(f(t_1, \ldots, t_p))]_1) && \text{by definition of } \bar{q} \\
&= q([f(s_X^n(t_1), \ldots, s_X^n(t_p))]_1) && \text{by definition of } s_X^n \\
&= h([f(q([s_X^n(t_1)]_1), \ldots, q([s_X^n(t_p)]_1))]_0) && \text{by Lemma 3.35 (1)} \\
&= h([f(\bar{q}([t_1]_{1+n}), \ldots, \bar{q}([t_p]_{1+n})]_0) && \text{by definition of } \bar{q},
\end{aligned}$$

     and similarly

$$\bar{q}([u]_{1+n}) = h([f(\bar{q}([u_1]_{1+n}), \ldots, \bar{q}([u_p]_{1+n})]_0).$$

     The equality then follows by inductive hypothesis.

     If $d(f) = 1$, then we get

$$\begin{aligned}
\bar{q}([t]_{1+n}) &= q([s_X^n(f(t_1, \ldots, t_p))]_1) && \text{by definition of } \bar{q} \\
&= q([f([t_1]_n, \ldots, [t_p]_n)]_1) && \text{by definition of } s_X^n \\
&= q([f([u_1]_n, \ldots, [u_p]_n)]_1) && \text{by assumption} \\
&= q([s_X^n(f(u_1, \ldots, u_p))]_1) && \text{by definition of } s_X^n \\
&= \bar{q}([u]_{1+n}) && \text{by definition of } \bar{q}.
\end{aligned}$$

- *For (cong$_{a.f}$) and (cong$_{\nu a.f}$):* Analogous to the above case.
- *For (ax$_{r=s}$):* We know that $Y \vdash_m r = s \in E$ with $t = (\tau r)\sigma$ and $u = (\tau s)\sigma$ for a derivably equivariant substitution $\sigma : Y \to \mathsf{Term}_{\Sigma,l}(X)$ and a permutation $\tau \in \mathsf{Perm}(\mathbb{A})$. Since we have assumed T to be depth-1, we have to consider two cases:

  If r and s are depth-0, then $l = 1 + n$. In this case, let $\bar{\sigma} : Y \to \mathsf{Term}_{\Sigma,0}(Q)$ be defined as
  $$\bar{\sigma}(x) = \bar{q}([\sigma(x)]_{1+n}) = q([s^n_X(\sigma(x))]_1).$$
  It then follows by Lemma 3.35 (4) that
  $$\bar{q}([t]_{1+n}) = q([s^n_X((\tau r)\sigma)]_1) = h([(\tau r)\bar{\sigma}]_0) \quad \text{and}$$
  $$\bar{q}([u]_{1+n}) = q([s^n_X((\tau s)\sigma)]_1) = h([(\tau s)\bar{\sigma}]_0).$$
  Finally, we can show $Q \vdash_0 (\tau r)\bar{\sigma} = (\tau s)\bar{\sigma}$ using (ax$_{r=s}$) by showing that $\bar{\sigma}$ is derivably equivariant: Let $\pi \in \mathsf{Perm}(\mathbb{A})$ and $x \in Y$. It then follows by inductive hypothesis that
  $$\begin{aligned} \pi\bar{\sigma}(x) &= \pi\bar{q}([\sigma(x)]_{1+n}) && \text{by definition of } \bar{\sigma} \\ &= \bar{q}([\pi\sigma(x)]_{1+n}) && \text{by (1)} \\ &= \bar{q}([\sigma(\pi x)]_{1+n}) && \text{by inductive hypothesis} \\ &= \bar{\sigma}(\pi x) && \text{by definition of } \bar{\sigma}. \end{aligned}$$
  If r and s are depth-1, then $l = n$. In this case, let $\bar{\sigma} : Y \to \mathsf{Term}_{\Sigma,0}(M_n(X))$ be defined as $\bar{\sigma}(x) = [\sigma(x)]_n$. It then follows by Lemma 3.35 (4) that
  $$\bar{q}([t]_{1+n}) = q([s^n_X((\tau r)\sigma)]_1) = q([(\tau r)\bar{\sigma}]_1) \quad \text{and}$$
  $$\bar{q}([u]_{1+n}) = q([s^n_X((\tau s)\sigma)]_1) = q([(\tau s)\bar{\sigma}]_1).$$
  We can then derive $Q \vdash_0 (\tau r)\bar{\sigma} = (\tau s)\bar{\sigma}$ by once again showing that $\bar{\sigma}$ is derivably equivariant: Let $\pi \in \mathsf{Perm}(\mathbb{A})$ and $x \in Y$. Then $\pi\bar{\sigma} = \pi[\sigma(x)]_n = [\pi\sigma(x)]_n = [\sigma(\pi x)]_n$ by Proposition 2.12 and by assumption.
- *For (perm$_f$):* We know that $t = \nu a.f(t_1, \dots, t_p)$ and $u = \nu b.f(u_1, \dots, u_p)$ where $a \neq b$, $a \# u_i$, and $X \vdash_m t_i = (a\ b)u_i$ is derivable for all $i \in \{1, \dots, p\}$. Since the operations are at most depth-1, we only have to consider two cases:

  If $d(f) = 0$, we get
  $$\bar{q}([t]_{1+n}) = h([\nu a.f(\bar{q}([t_1]_{1+n}), \dots, \bar{q}([t_p]_{1+n}))]_0) \quad \text{and}$$
  $$\bar{q}([u]_{1+n}) = h([\nu b.f(\bar{q}([u_1]_{1+n}), \dots, \bar{q}([u_p]_{1+n}))]_0)$$
  analogously to (cong$_f$). Furthermore, for $i \in \{1, \dots, p\}$, we have
  $$\begin{aligned} \bar{q}([t_i]_{1+n}) &= \bar{q}([(a\ b)u_i]_{1+n} && \text{by inductive hypothesis} \\ &= (a\ b)\bar{q}([u_i]_{1+n}) && \text{by (1),} \end{aligned}$$
  and also $a \notin \mathsf{supp}(\bar{q}([u_i]_{1+n})) \subseteq \mathsf{supp}(u_i)$ by (1). Thus, the equality follows by (perm).

  If $d(f) = 1$, we get
  $$\bar{q}([t]_{1+n}) = \bar{q}([\nu a.f([t_1]_n, \dots, [t_p]_n)]_1) \quad \text{and}$$
  $$\bar{q}([u]_{1+n}) = \bar{q}([\nu b.f([u_1]_n, \dots, [u_p]_n)]_1)$$
  analogously to (cong$_f$). Furthermore, for $i \in \{1, \dots, p\}$, we have $[t_i]_n = [(a\ b)u_i]_n = (a\ b)[u_i]_n$ and $a \notin \mathsf{supp}([u_i]_n) \subseteq \mathsf{supp}(u_i)$ by assumption and Proposition 2.12. The result then follows by (perm).

3. $\bar{q} : (M_{1+n}(X), \mu_X^{0,1+n}) \to (Q, h)$ is indeed a morphism between M_0-algebras:

   First, note that equivariance follows from (1) and Proposition 2.12.

   We will also have to show that $h \circ M_0(\bar{q}) = \bar{q} \circ \mu_X^{0,1+n}$. So let $[t]_0 \in M_0(M_{1+n}(X))$.

   Fix a splitting $\sigma : M_{1+n}(X) \to \mathsf{Term}_{\Sigma,1+n}(X)$, in that $[\cdot]_{1+n} \circ \sigma = \mathsf{id}$. Let $\bar{\sigma} : M_{1+n}(X) \to \mathsf{Term}_{\Sigma,0}(Q)$ be defined as $\bar{\sigma}(x) = \bar{q}(x) = \bar{q}([\sigma(x)]_{1+n}) = q([s_X^n(\sigma(x))]_1)$.

   It then follows that

$$\begin{aligned} h(M_0(\bar{q})([t]_0)) &= h([t\bar{\sigma}]_0) && \text{by Definition 3.28 because } \sigma_{\bar{q}} = \bar{\sigma} \\ &= q([s_X^n(t\sigma)]_1) && \text{by Lemma 3.35 (3)} \\ &= \bar{q}([t\sigma]_{1+n}) && \text{by definition of } \bar{q} \\ &= \bar{q}(\mu_X^{0,1+n}([t]_0)) && \text{by Lemma 3.30.} \end{aligned}$$

4. $\bar{q}$ satisfies the given property; i.e., $\bar{q} \circ \mu_X^{1,n} = q$. So let $[t]_1 \in M_1(M_n(X))$.

   We fix a splitting $\sigma : M_n(X) \to \mathsf{Term}_{\Sigma,n}(X)$, in that $[\cdot]_n \circ \sigma = \mathsf{id}$. Let $\bar{\sigma} : M_n(X) \to \mathsf{Term}_{\Sigma,0}(M_n(X))$ be defined as $\bar{\sigma}(x) = [\sigma(x)]_1 = x$.

   It then follows that

$$\begin{aligned} \bar{q}(\mu_X^{1,n}([t]_1)) &= \bar{q}([t\sigma]_{1+n}) && \text{by Lemma 3.30} \\ &= q([s_X^n(t\sigma)]_1) && \text{by definition of } \bar{q} \\ &= q([t\bar{\sigma}]_1) && \text{by Lemma 3.35 (4)} \\ &= q(M_1(\mathsf{id})([t]_1)) && \text{by Definition 3.28 because } \sigma_{\mathsf{id}} = \bar{\sigma} \\ &= q([t]_1) && \text{by functoriality of } M_1. \end{aligned}$$

5. If there is another such morphism q', then $q'([t]_1) = \bar{q}([t]_1)$ for every $t \in \mathsf{Term}_{\Sigma,1}(M_n(X))$.

   So assume that $q' : M_{1+n}(X) \to Q$ is a morphism between M_0-algebras such that

$$q' \circ \mu_X^{1,n} = q. \tag{3.5}$$

   We will proceed by induction on t, only considering cases where t has at least uniform depth 1.

   - *For $t = f(t_1, \ldots, t_p)$ with $f/p \in \Sigma_0$, $t_1, \ldots, t_p \in \mathsf{Term}_{\Sigma,m}(X)$, and $1 + n = m + d(f)$:* Since all operations are at most depth-1, we only have to consider two cases:

     If $d(f) = 0$, then

$$\begin{aligned} & q'([f(t_1, \ldots, t_p)]_{1+n}) \\ = \; & q'(\mu^{0,1+n}([f([t_1]_1, \ldots, [t_p]_1)]_0)) && \text{by Lemma 3.32} \\ = \; & h(M_0(q')([f([t_1]_1, \ldots, [t_p]_1)]_0)) && \text{because } q' \text{ is a morphism between } M_0\text{-algebras} \\ = \; & h([f(q'([t_1]_1), \ldots, q'([t_p]_1)]_0) && \text{by Definition 3.28.} \end{aligned}$$

     With a similar argument, we get

$$\bar{q}([f(t_1, \ldots, t_p)]_{1+n}) = h([f(\bar{q}([t_1]_1), \ldots, \bar{q}([t_p]_1))]_0)$$

     The equality then follows by inductive hypothesis.

If $d(f) = 1$, then

$$\begin{aligned}
q'([f(t_1, \ldots, t_p)]_{1+n}) &= q'(\mu_X^{1,n}([f([t_1]_n, \ldots, [t_p]_n)]_1)) && \text{by Lemma 3.32} \\
&= q([f([t_1]_n, \ldots, [t_p]_n)]_1) && \text{by Equation 3.5} \\
&= q([s_X^n(f(t_1, \ldots, t_p)]_1) && \text{by definition of } s_X^n \\
&= \bar{q}([f(t_1, \ldots, t_p)]_{1+n}) && \text{by definition of } \bar{q}.
\end{aligned}$$

- *For $t = a.f(t_1, \ldots, t_p)$ and $t = \nu a.f(t_1, \ldots, t_p)$:* Analogous to the above case.

□

4 Describing Local Freshness Semantics

In the following chapter, we will describe a graded theory to capture the local-freshness semantics of nominal automata. To prove this, we will show that the interpretation of pretrace terms modulo derivable equality under local freshness semantics can be expressed as an injective morphism from the free model generated by the theory.

Definition 4.1. We define a signature Σ with $\Sigma_0 = \{+/2, \bot/0\}$, $\Sigma_\mathsf{f} = \{\mathsf{pre}/1\}$, and $\Sigma_\mathsf{b} = \{\mathsf{abs}/1\}$, where all pure operations are depth-0 and all free and bound operations are depth-1. We abbreviate

$$at \coloneqq a.pre(t) \quad \text{and} \quad |at \coloneqq \nu a.abs(t).$$

The graded theory T is then defined over this signature with the axioms

$$X \vdash_0 x + y = y + x, \tag{4.1}$$
$$X \vdash_0 (x + y) + z = x + (y + z), \tag{4.2}$$
$$X \vdash_0 x + x = x, \tag{4.3}$$
$$X \vdash_0 x + \bot = x, \tag{4.4}$$
$$X \vdash_1 a(x + y) = ax + ay, \tag{4.5}$$
$$X \vdash_1 |a(x + y) = |ax + |ay, \tag{4.6}$$
$$X \vdash_1 a\bot = \bot, \tag{4.7}$$
$$X \vdash_1 |a\bot = \bot, \tag{4.8}$$
$$X \vdash_1 |ax = |ax + ax, \tag{4.9}$$

ranging over all $a \in \mathbb{A}$, all nominal sets X, and all elements $x, y, z \in X$.

The syntax clearly hints at that of bar strings from RNNAs, with the addition of a poststate, indicating that we can describe the pretraces generated by a state of an RNNA as a term and thus describe its semantics. For example, we will show that one can express the semantics of s_0 in the RNNA given in Example 2.21 at depth 2 using the pretrace term $|bbs_2 + |bas_3$ in a well-defined manner.

4.1 Words and Alpha-Equivalence

We will first restrict ourselves to the pretraces of single words and define a notion of local freshness semantics for them.

Definition 4.2 (Pretraces)**.** A **pretrace** of depth $n \in \mathbb{N}_0$ over a nominal set X of variables is a term $t \in \mathsf{Term}_{\Sigma,n}(X)$ which does not contain any occurrences of $+$ or $\bot$.

Pretraces over X of depth n are obviously isomorphic to $\bar{\mathbb{A}}^n \times X$, as already suggested by the syntax of terms. In the following, we will identify the two sets.

For nominal automata under local freshness semantics, the language consists of all words alpha-equivalent to words in the literal language, without the bars. To define something similar for

$$\text{(refl)}\frac{}{x \mathrel{\hat{\equiv}}_\alpha x} \qquad \text{(symm)}\frac{u \mathrel{\hat{\equiv}}_\alpha t}{t \mathrel{\hat{\equiv}}_\alpha u} \qquad \text{(trans)}\frac{t \mathrel{\hat{\equiv}}_\alpha v \quad v \mathrel{\hat{\equiv}}_\alpha u}{t \mathrel{\hat{\equiv}}_\alpha u}$$

$$\text{(cong}_a\text{)}\frac{t \mathrel{\hat{\equiv}}_\alpha u}{at \mathrel{\hat{\equiv}}_\alpha au} \qquad \text{(cong}_{|a}\text{)}\frac{t \mathrel{\hat{\equiv}}_\alpha u}{|at \mathrel{\hat{\equiv}}_\alpha |au} \quad (a \in \mathbb{A})$$

$$\text{(perm)}\frac{a\#u \quad t \mathrel{\hat{\equiv}}_\alpha (a\ b)u}{|at \mathrel{\hat{\equiv}}_\alpha |bu} \quad (a, b \in \mathbb{A}, a \neq b)$$

Figure 4.1: System of rules for deriving alpha-equivalence for pretraces.

our graded theory, we will first need to define a notion of alpha-equivalence, taking into account the poststate a pretrace ends with. We will first define this as a subset of the derivation rules for derivable equality:

Definition 4.3 (Alpha-Equivalence on Pretraces)**.** Let **alpha-equivalence** on two pretraces be the relation $\hat{\equiv}_\alpha$ inductively defined by Figure 4.1.

Lemma 4.4. *Let $t, u \in \bar{\mathbb{A}}^n \times X$ be pretraces over X. If $t \mathrel{\hat{\equiv}}_\alpha u$, then $X \vdash_n t = u$ is derivable.*

Proof. This follows from the fact that the derivation rules are a subset of those used for derivable equality. □

Lemma 4.5. *The relation $\hat{\equiv}_\alpha$ is an equivariant equivalence-relation.*

Proof. Completely analogous to Proposition 3.11 and Lemma 3.18. □

With this, we can partition the set $\bar{\mathbb{A}}^n \times X$ of pretraces into equivalence classes. We will denote the equivalence class of a pretrace t as $[t]_{\hat{\alpha}}$.

This is enough to define the local freshness semantics of a pretrace:

Definition 4.6. We define the **local freshness semantics of a pretrace** using the function $\hat{D}_n : (\bar{\mathbb{A}}^n \times X)/\hat{\equiv}_\alpha \to \mathbb{A}^n \times X$ with

$$\hat{D}_n([t]_{\hat{\alpha}}) = \{\hat{\mathsf{ub}}_n(w) : w \in [t]_{\hat{\alpha}}\},$$

where $\hat{\mathsf{ub}}_n : \bar{\mathbb{A}}^n \times X \to \mathbb{A}^n \times X$ removes the bars from a word and is defined inductively by

$$\begin{aligned} \hat{\mathsf{ub}}_0(x) &= x, \\ \hat{\mathsf{ub}}_{n+1}(at) &= (a, \hat{\mathsf{ub}}_n(t)), \\ \hat{\mathsf{ub}}_{n+1}(|at) &= (a, \hat{\mathsf{ub}}_n(t)). \end{aligned}$$

When the depth of the term is not important, we often just write $\hat{D}(t)$ or $\hat{\mathsf{ub}}(t)$.

Lemma 4.7. *The above description for $\hat{D}_n$ is equivariant for all $n \in \mathbb{N}_0$.*

Proof. First note that $\hat{\mathsf{ub}}_n$ is equivariant: We will show this by induction on t. Let $\pi \in \mathsf{Perm}(\mathbb{A})$.

- *For $t = x$ with $x \in X$:* We have $\hat{\mathsf{ub}}_0(\pi x) = \pi x = \pi\hat{\mathsf{ub}}_0(x)$.
- *For $t = at'$ with $t' \in \bar{\mathbb{A}}^{n'} \times X, n = n' + 1$:* We have

$$\begin{aligned}
\hat{\mathsf{ub}}_n((\pi a)(\pi t')) &= (\pi a, \hat{\mathsf{ub}}_{n'}(\pi t')) && \text{by definition}\\
&= (\pi a, \pi\hat{\mathsf{ub}}_{n'}(t')) && \text{by inductive hypothesis}\\
&= \pi(a, \hat{\mathsf{ub}}_{n'}(t')) && \text{by definition}\\
&= \pi\hat{\mathsf{ub}}_n(at') && \text{by definition.}
\end{aligned}$$

- *For $t = {|}at'$ with $t' \in \bar{\mathbb{A}}^{n'} \times X, n = n' + 1$:* Analogous to the above case.

We can conclude that $\hat{D}_n$ is also equivariant: Note that

$$\hat{D}_n([t]_{\hat{\alpha}}) = \hat{\mathsf{ub}}_n\left[[t]_{\hat{\alpha}}\right],$$

where $\hat{\mathsf{ub}}_n[\cdot]$ is the direct image under $\hat{\mathsf{ub}}_n$. Equivariance then follows from Proposition 2.11. □

It is important to note that, since we take into account the poststate for alpha-equivalence, the language generated by a trace under local freshness semantics is not the same as taking the local freshness semantics of a pretrace and then removing poststates:

Example 4.8. Consider the pretrace ${|}ab \in \bar{\mathbb{A}}^1 \times \mathbb{A}$ and the corresponding trace ${|}a \in \bar{\mathbb{A}}^1$.

By Definition 4.6, we have

$$\hat{D}([{|}ab]_1) = \{cb : c \in \mathbb{A}, c \neq b\},$$

however, by Definition 2.20,

$$D(\{[{|}a]\}) = \mathbb{A}.$$

To make this dependency on the poststate more clear, we will proceed to give an explicit characterization of the support of a pretrace modulo alpha-equivalence.

4.1.1 Free Names

First, we will define the free names of a pretrace explicitly:

Definition 4.9 (Free Names)**.** Given a pretrace $t \in \bar{\mathbb{A}}^n \times X$, we define the **free names** $\mathsf{FN}_n(t)$ in the term inductively as

$$\begin{aligned}
\mathsf{FN}_0(x) &= \mathsf{supp}_X(x),\\
\mathsf{FN}_{n+1}(at) &= \mathsf{FN}_n(t) \cup \{a\},\\
\mathsf{FN}_{n+1}({|}at) &= \mathsf{FN}_n(t) \setminus \{a\}.
\end{aligned}$$

When the depth of the term is not important, we often just write $\mathsf{FN}(t)$.

We will also show some useful statements regarding these free names:

Lemma 4.10. *The function $\mathsf{FN}_n : \bar{\mathbb{A}}^n \times X \to \mathcal{P}(\mathbb{A})$ is equivariant for every $n \in \mathbb{N}_0$.*

Proof. By induction on n. Let $\pi \in \mathsf{Perm}(\mathbb{A})$ and $t \in \bar{\mathbb{A}}^n \times X$.

- *For $t = x$ with $x \in X$:* By equivariance of supp, we get $\mathsf{FN}_0(\pi x) = \mathsf{supp}(\pi x) = \pi\mathsf{supp}(x) = \pi\mathsf{FN}_0(x)$.
- *For $t = at'$ with $t' \in \bar{\mathbb{A}}^{n'}, n = n' + 1$:*

$$\begin{aligned}
\mathsf{FN}_n(\pi t) &= \mathsf{FN}_{n'}(\pi t') \cup \{\pi(a)\} && \text{by definition} \\
&= \pi\mathsf{FN}_{n'}(t') \cup \pi\{a\} && \text{by inductive hypothesis} \\
&= \pi(\mathsf{FN}_{n'}(t') \cup \{a\}) && \text{by Proposition 2.10} \\
&= \pi\mathsf{FN}_n(t) && \text{by definition.}
\end{aligned}$$

- *For $t = \mathsf{l}at'$:* Analogous to the above case.

□

Lemma 4.11. *Let $t, u \in \bar{\mathbb{A}}^n \times X$ be pretraces over X. If $t \mathrel{\hat{\equiv}_\alpha} u$, then $\mathsf{FN}(t) = \mathsf{FN}(u)$.*

Proof. By induction on the derivation of $t \mathrel{\hat{\equiv}_\alpha} u$.

- *For (refl):* We know that $t = u = x$ for $x \in X$. By definition, we have $\mathsf{FN}(t) = \mathsf{supp}_X(x) = \mathsf{FN}(u)$.
- *For (symm):* We know that $u \mathrel{\hat{\equiv}_\alpha} t$ is derivable. By inductive hypothesis, we get $\mathsf{FN}(u) = \mathsf{FN}(t)$.
- *For (trans):* We know that, for some $v \in \bar{\mathbb{A}}^n \times X$, $t \mathrel{\hat{\equiv}_\alpha} v$ and $v \mathrel{\hat{\equiv}_\alpha} u$ are derivable. By inductive hypothesis, we get $\mathsf{FN}(t) = \mathsf{FN}(v) = \mathsf{FN}(u)$.
- *For (cong$_a$):* We know that $t = at'$, $u = au'$, and $t' \mathrel{\hat{\equiv}_\alpha} u'$ is derivable. By inductive hypothesis, we get $\mathsf{FN}(t') = \mathsf{FN}(u')$ and thus $\mathsf{FN}(t) = \mathsf{FN}(t') \cup \{a\} = \mathsf{FN}(u') \cup \{a\} = \mathsf{FN}(u)$.
- *For (cong$_{\mathsf{l}a}$):* We know that $t = \mathsf{l}at'$, $u = \mathsf{l}au'$, and $t' \mathrel{\hat{\equiv}_\alpha} u'$ is derivable. By inductive hypothesis, we get $\mathsf{FN}(t') = \mathsf{FN}(u')$ and thus $\mathsf{FN}(t) = \mathsf{FN}(t') \setminus \{a\} = \mathsf{FN}(u') \setminus \{a\} = \mathsf{FN}(u)$.
- *For (perm):* We know that $t = \mathsf{l}at'$, $u = \mathsf{l}bu'$, and $t' \mathrel{\hat{\equiv}_\alpha} (a\ b)u$ is derivable with $a \neq b$ and $a \# u'$. By inductive hypothesis, we know that $\mathsf{FN}(t') = \mathsf{FN}((a\ b)u')$. It follows from Lemma 4.10 that $\mathsf{FN}(t') = (a\ b)\mathsf{FN}(u')$. We will consider three cases for atoms $c \in \mathbb{A}$:

  For $c = a$: By definition, $a \notin \mathsf{FN}(t) = \mathsf{FN}(t') \setminus \{a\}$. Furthermore, we have $a \# u'$ and thus $a \notin \mathsf{FN}(u') \subseteq \mathsf{supp}(u')$. It follows that $a \notin \mathsf{FN}(u) = \mathsf{FN}(u') \setminus \{b\}$.

  For $c = b$: By definition, $b \notin \mathsf{FN}(u) = \mathsf{FN}(u') \setminus \{b\}$. Furthermore, it follows from $a \notin \mathsf{FN}(u')$ that $b \notin (a\ b)\mathsf{FN}(u')$ and thus $b \notin \mathsf{FN}(t) = ((a\ b)\mathsf{FN}(u')) \setminus \{a\}$.

  For $c \neq a$ and $c \neq b$:

$$c \in \mathsf{FN}(t) = ((a\ b)\mathsf{FN}(u')) \setminus \{a\} \quad \Leftrightarrow \quad c \in \mathsf{FN}(u') \quad \Leftrightarrow \quad c \in \mathsf{FN}(u) = \mathsf{FN}(u') \setminus \{b\}.$$

□

We will also show that, if an atom does not occur freely in a pretrace, there is an alpha-equivalent pretrace where the atom does not occur at all.

Lemma 4.12. *Let $t \in \bar{\mathbb{A}}^n \times X$ be a pretrace over X and $N \in \mathcal{P}_{\mathsf{f}}(\mathbb{A})$ a finite set of atoms. If $N \cap \mathsf{FN}(t) = \emptyset$, then there exists a pretrace $\tilde{t} \in \bar{\mathbb{A}}^n \times X$ with $t \mathrel{\hat{\equiv}_\alpha} \tilde{t}$ and $N \cap \mathsf{supp}(\tilde{t}) = \emptyset$.*

Proof. By induction on n.

- *For $t = x$ with $x \in X$:* Set $\tilde{t} = x$. By (refl), we have $t \mathrel{\hat{\equiv}_\alpha} \tilde{t}$. Furthermore, we get $N \cap \mathsf{supp}(\tilde{t}) = N \cap \mathsf{FN}(t) = \emptyset$ by Definition 4.9.
- *For $t = at'$ with $t' \in \bar{\mathbb{A}}^{n'} \times X, n = n' + 1$:* Since $\mathsf{FN}(t') \subseteq \mathsf{FN}(t') \cup \{a\} = \mathsf{FN}(t)$, we know that $N \cap \mathsf{FN}(t') = \emptyset$. By inductive hypothesis, we get a pretrace $\tilde{t}'$ with $t' \mathrel{\hat{\equiv}_\alpha} \tilde{t}'$ and $N \cap \mathsf{supp}(\tilde{t}') = \emptyset$. We set $\tilde{t} = a\tilde{t}'$.

  It follows from (cong_a) that $t = at' \mathrel{\hat{\equiv}_\alpha} a\tilde{t}' = \tilde{t}$ and, since $a \notin N$, we get

  $$N \cap \mathsf{supp}(\tilde{t}) = (N \cap \mathsf{supp}(\tilde{t}')) \cup (N \cap \{a\}) = \emptyset.$$

- *For $t = \mathsf{I}at'$ with $t' \in \bar{\mathbb{A}}^{n'} \times X, n = n' + 1$:* Pick any $b \in \mathbb{A} \setminus N$ with $b\#t'$. By applying (perm), we get $t \mathrel{\hat{\equiv}_\alpha} \mathsf{I}b((a\ b)t')$.

  We will show that $N \cap \mathsf{FN}((a\ b)t') = N \cap (a\ b)\mathsf{FN}(t') = \emptyset$: Let $c \in N$. If $c = a$, then $a \notin (a\ b)\mathsf{FN}(t')$ because $b \notin \mathsf{FN}(t') \subseteq \mathsf{supp}(t')$. Otherwise, $c \notin \mathsf{FN}(t')$. Since $b \notin N$ and $c \in N$, we have $b \neq c$ and thus $c \notin (a\ c)\mathsf{FN}(t')$.

  By inductive hypothesis, we get a pretrace $\tilde{t}'$ with $(a\ b)t' \mathrel{\hat{\equiv}_\alpha} \tilde{t}'$ and $N \cap \mathsf{supp}(\tilde{t}') = \emptyset$. We set $\tilde{t} = \mathsf{I}b\tilde{t}'$.

  By applying ($\text{cong}_{\mathsf{I}b}$) and (trans), we get $t \mathrel{\hat{\equiv}_\alpha} \mathsf{I}b((a\ b)t') \mathrel{\hat{\equiv}_\alpha} b\tilde{t}' = \tilde{t}$. Finally, since $b \notin N$, we get

  $$N \cap \mathsf{supp}(\tilde{t}) = (N \cap \mathsf{supp}(\tilde{t}')) \cup (N \cap \{b\}) = \emptyset.$$

□

4.1.2 An Alternative Definition of Alpha-Equivalence

The derivation rules make it easier to reason about pretraces as a fragment of regular terms. However, we will prefer to use an alternative inductive characterization of alpha-equivalence:

Proposition 4.13. *Let $t, u \in \bar{\mathbb{A}}^n \times X$ be pretraces over X. Then $t \mathrel{\hat{\equiv}_\alpha} u$ is derivable iff either*

1. *$t = u = x$ for $x \in X$,*
2. *$t = at'$ and $u = au'$ with $t' \mathrel{\hat{\equiv}_\alpha} u'$,*
3. *$t = \mathsf{I}at'$ and $u = \mathsf{I}au'$ with $t' \mathrel{\hat{\equiv}_\alpha} u'$, or*
4. *$t = \mathsf{I}at'$ and $u = \mathsf{I}bu'$ with $a \neq b$, $a \notin \mathsf{FN}(u')$, and $t' \mathrel{\hat{\equiv}_\alpha} (a\ b)u'$.*

Proof. 'If': We will show the statement for each case individually.

1. *If $t = u = x$ with $x \in X$:* Then we can derive $x \mathrel{\hat{\equiv}_\alpha} x$ by (refl).
2. *If $t = at'$ and $u = au'$ with $t' \mathrel{\hat{\equiv}_\alpha} u'$:* Then we can derive $at' \mathrel{\hat{\equiv}_\alpha} au'$ by (cong_a).
3. *If $t = \mathsf{I}at'$ and $u = \mathsf{I}au'$ with $t' \mathrel{\hat{\equiv}_\alpha} u'$:* Then we can derive $\mathsf{I}at' \mathrel{\hat{\equiv}_\alpha} \mathsf{I}au'$ by ($\text{cong}_{\mathsf{I}a}$).
4. *If $t = \mathsf{I}at'$ and $u = \mathsf{I}bu'$ with $a \neq b$, $a \notin \mathsf{FN}(u')$, and $t' \mathrel{\hat{\equiv}_\alpha} (a\ b)u'$:* By applying Lemma 4.12 with $N = \{a\}$, we get a pretrace $\tilde{u}$ with $a\#\tilde{u}'$ and $u' \mathrel{\hat{\equiv}_\alpha} \tilde{u}'$ (and, by Lemma 4.5, $(a\ b)u' \mathrel{\hat{\equiv}_\alpha} (a\ b)\tilde{u}'$). It then follows that

   $$\begin{aligned} \mathsf{I}at' &\mathrel{\hat{\equiv}_\alpha} \mathsf{I}b\tilde{u}' && \text{by applying (perm)} \\ &\mathrel{\hat{\equiv}_\alpha} \mathsf{I}bu' && \text{by applying (cong}_{\mathsf{I}b}\text{).} \end{aligned}$$

'Only if': By induction on the derivation of $t \mathrel{\hat{\equiv}_\alpha} u$.

- *For (refl):* We know that $t = u = x$ for $x \in X$. Thus, (1) follows immediately.
- *For (symm):* We know that $u \mathrel{\hat{\equiv}_\alpha} t$ is derivable and, by inductive hypothesis, we only need to consider the following cases:
  1. *If $u = t = x$ with $x \in X$:* Then (1) follows immediately.
  2. *If $u = au'$ and $t = at'$ with $u' \mathrel{\hat{\equiv}_\alpha} t'$:* Then, by (symm), we get $t' \mathrel{\hat{\equiv}_\alpha} u'$ and (2) follows immediately.
  3. *If $u = \mathsf{l}au'$ and $t = \mathsf{l}at'$ with $u' \mathrel{\hat{\equiv}_\alpha} t'$:* Then, by (symm), we get $t' \mathrel{\hat{\equiv}_\alpha} u'$ and (3) follows immediately.
  4. *If $u = \mathsf{l}bu'$ and $t = \mathsf{l}at'$ with $a \neq b$, $b \notin \mathsf{FN}(t')$, and $u' \mathrel{\hat{\equiv}_\alpha} (a\ b)t'$:* By Lemma 4.5, we know that $(a\ b)u' \mathrel{\hat{\equiv}_\alpha} t'$ and, by (symm), $t' \mathrel{\hat{\equiv}_\alpha} (a\ b)u'$. Additionally, we have

$$
\begin{aligned}
b \notin \mathsf{FN}(t') \quad &\Longrightarrow \quad b \notin \mathsf{FN}((a\ b)u') && \text{by Lemma 4.11} \\
&\Longrightarrow \quad b \notin (a\ b)\mathsf{FN}(u') && \text{by Lemma 4.10} \\
&\Longrightarrow \quad a \notin \mathsf{FN}(u').
\end{aligned}
$$

     Thus, we can conclude (4).
- *For (trans):* We know that, for some $v \in \bar{\mathbb{A}}^n \times X$, $t \mathrel{\hat{\equiv}_\alpha} v$ and $v \mathrel{\hat{\equiv}_\alpha} u$ are derivable. By inductive hypothesis, we only need to consider the following cases:
  1. *If $t = v = x$ with $x \in X$:* Then, by inductive hypothesis, $v = u = x$. Thus, (1) follows immediately.
  2. *If $t = at'$ and $v = av'$ with $t' \mathrel{\hat{\equiv}_\alpha} v'$:* It follows by inductive hypothesis that $u = au'$ with $v' \mathrel{\hat{\equiv}_\alpha} u'$. By (trans), we can conclude that $t' \mathrel{\hat{\equiv}_\alpha} u'$.
  3. *If $t = \mathsf{l}at'$ and $v = \mathsf{l}av'$ with $t' \mathrel{\hat{\equiv}_\alpha} v'$:* By inductive hypothesis, we need to consider the following cases for u:

     For $u = \mathsf{l}au'$ with $v' \mathrel{\hat{\equiv}_\alpha} u'$, we can conclude $t' \mathrel{\hat{\equiv}_\alpha} u'$ by applying (trans) and thus, (3) holds.

     For $u = \mathsf{l}bu'$ with $a \neq b$, $a \notin \mathsf{FN}(u')$, and $v' \mathrel{\hat{\equiv}_\alpha} (a\ b)u'$, we can conclude $t' \mathrel{\hat{\equiv}_\alpha} (a\ b)u'$ by applying (trans). Thus, (4) holds.
  4. *If $t = \mathsf{l}at'$ and $v = \mathsf{l}bv'$ with $a \neq b$, $a \notin \mathsf{FN}(v')$, and $t' \mathrel{\hat{\equiv}_\alpha} (a\ b)v'$:* By inductive hypothesis, we need to consider the following cases for u:

     For $u = \mathsf{l}bu'$ with $v' \mathrel{\hat{\equiv}_\alpha} u'$, we can conclude from Lemma 4.5 that $(a\ b)v' \mathrel{\hat{\equiv}_\alpha} (a\ b)u'$ and, by applying (trans), $t' \mathrel{\hat{\equiv}_\alpha} (a\ b)u'$. Furthermore, it follows from Lemma 4.11 that $a \notin \mathsf{FN}(u') = \mathsf{FN}(t')$, so (4) holds.

     For $u = \mathsf{l}au'$ with $a \neq b$, $b \notin \mathsf{FN}(u')$, and $v' \mathrel{\hat{\equiv}_\alpha} (a\ b)u'$, we can conclude from Lemma 4.5 that $(a\ b)v' \mathrel{\hat{\equiv}_\alpha} u'$ and, by applying (trans), we get $t' \mathrel{\hat{\equiv}_\alpha} u'$. Thus, (3) holds.

     For $u = \mathsf{l}cu'$ with $c \neq a$, $c \neq b$, $b \notin \mathsf{FN}(u')$, and $v' = (b\ c)u'$, we can first conclude $a \notin \mathsf{FN}(u') = \mathsf{FN}((b\ c)v') = (b\ c)\mathsf{FN}(v')$ from Lemma 4.11 and Lemma 4.10 because $a \notin \mathsf{FN}(v')$. By Lemma 4.12 with $N = \{a, b\}$, there exists a pretrace $\tilde{u}'$ with $u' \mathrel{\hat{\equiv}_\alpha} \tilde{u}'$

and $a, b\#\tilde{u}'$. Finally, (4) follows from (trans) and

$$\begin{aligned}
t' &\hat{\equiv}_\alpha (a\ b)v' && \text{by assumption}\\
&\hat{\equiv}_\alpha (a\ b)(b\ c)u' && \text{by Lemma 4.5 and } v' \hat{\equiv}_\alpha (b\ c)u'\\
&\hat{\equiv}_\alpha (a\ b)(b\ c)\tilde{u}' && \text{by Lemma 4.5 and } u' \hat{\equiv}_\alpha \tilde{u}'\\
&\hat{\equiv}_\alpha (a\ c)\tilde{u}' && \text{by Proposition 2.5, } (a\ b)(b\ c)(c) = a = (a\ c)(c)\text{, and } a, b\#\tilde{u}'\\
&\hat{\equiv}_\alpha (a\ c)u' && \text{by Lemma 4.5 and } u' \hat{\equiv}_\alpha \tilde{u}'.
\end{aligned}$$

- *For (cong$_a$):* We know that $t = at'$, $u = au'$, and $t' \hat{\equiv}_\alpha u'$ is derivable. Thus, we can conclude (2).
- *For (cong$_{\mathsf{I}a}$):* We know that $t = \mathsf{I}at'$, $u = \mathsf{I}au'$, and $t' \hat{\equiv}_\alpha u$ is derivable. Thus, we can conclude (3).
- *For (perm):* We know that $t = \mathsf{I}at'$, $u = \mathsf{I}bu'$, and $t' \hat{\equiv}_\alpha (a\ b)u'$ is derivable with $a \neq b$ and $a\#u'$. Then $a \notin \mathsf{FN}(u') \subseteq \mathsf{supp}(u')$ and we can conclude (4).

□

With this characterization of alpha-equivalence, we can now show that the "free names" are exactly the support of the equivalence classes of pretraces modulo alpha-equivalence.

Proposition 4.14. *If $t \in \bar{\mathbb{A}}^n \times X$ is a pretrace over X, then $\mathsf{supp}([t]_{\hat{\alpha}}) = \mathsf{FN}(t)$.*

Proof. We will first show that $\mathsf{FN}(t)$ is a support for $[t]_{\hat{\alpha}}$, and thus $\mathsf{supp}([t]_{\hat{\alpha}}) \subseteq \mathsf{FN}(t)$. We will use Proposition 2.13 by showing, by induction on t, that $\pi t \hat{\equiv}_\alpha t$ for all $\pi \in \mathsf{Fix}(\mathsf{FN}(t))$.

- *For $t = x$ with $x \in X$:* Then $\pi \in \mathsf{Fix}(\mathsf{FN}(t)) = \mathsf{Fix}(\mathsf{supp}(x))$ and thus $\pi x = x \hat{\equiv}_\alpha x$ by Proposition 4.13 (1).
- *For $t = at'$ with $t' \in \bar{\mathbb{A}}^{n'} \times X, n = n' + 1$:* Then we have $\mathsf{FN}(t') \subseteq \mathsf{FN}(t') \cup \{a\} = \mathsf{FN}(t)$ and, by inductive hypothesis, $\pi \in \mathsf{Fix}(\mathsf{FN}(t)) \subseteq \mathsf{Fix}(\mathsf{FN}(t')) \subseteq \mathsf{Fix}(\mathsf{supp}([t']_{\hat{\alpha}}))$. It follows from Proposition 2.12 that $[\pi t']_{\hat{\alpha}} = \pi[t']_{\hat{\alpha}} = [t']_{\hat{\alpha}}$. Thus, we get

$$\begin{aligned}
\pi(at') &= (\pi a)(\pi t') && \text{by definition}\\
&= a(\pi t') && \text{because } a \in \mathsf{FN}(t)\\
&\hat{\equiv}_\alpha at' && \text{by Proposition 4.13 (2) and } \pi t' \hat{\equiv}_\alpha t'.
\end{aligned}$$

- *For $t = \mathsf{I}at'$ with $t' \in \bar{\mathbb{A}}^{n'} \times X, n = n' + 1$:* We know that $\mathsf{FN}(t) = \mathsf{FN}(t') \setminus \{a\}$.

  If $\pi(a) = a$, then $\pi \in \mathsf{Fix}(\mathsf{FN}(t'))$ and we get $\pi(\mathsf{I}at') \hat{\equiv}_\alpha \mathsf{I}at'$ analogously to the above case.

  If $\pi(a) \neq a$, then we know that $\pi(a) \notin \mathsf{FN}(t')$: If we assume $\pi(a) \in \mathsf{FN}(t')$ and (by $\pi(a) \neq a$) also $\pi(a) \in \mathsf{FN}(t') \setminus \{a\} = \mathsf{FN}(t)$, then $\pi(\pi(a)) = \pi(a)$ because $\pi \in \mathsf{Fix}(\mathsf{FN}(t))$. However, since $\pi(a) \neq a$ and π is injective, we have $\pi(\pi(a)) \neq \pi(a)$.

  Furthermore, since

$$\begin{aligned}
(a\ \pi(a))(a) &= \pi(a), && \text{and}\\
(a\ \pi(a))(b) &= b = \pi(b) && \forall b \in \mathsf{FN}(t') \setminus \{a\},
\end{aligned}$$

  we can conclude that $(a\ \pi(a))t' = \pi t'$ by inductive hypothesis and Proposition 2.5. It then follows from Proposition 4.13 (4) and (3) that

$$\mathsf{I}at' \hat{\equiv}_\alpha \mathsf{I}(\pi a)((a\ \pi(a))t') \hat{\equiv}_\alpha \mathsf{I}(\pi a)(\pi t') = \pi(\mathsf{I}at').$$

Conversely, we will now show that $\mathsf{FN}(t) \subseteq \mathsf{supp}([t]_{\hat{\alpha}})$. It is enough to show that $\mathsf{FN}(t) \subseteq \mathsf{supp}(u)$ for all pretraces $u \in \bar{\mathbb{A}}^n \times X$ with $u \mathrel{\hat{\equiv}_\alpha} t$ by Proposition 2.14.

- *For $t = x$ with $x \in X$:* It follows from Proposition 4.13 that $u = x$ and thus $\mathsf{FN}(t) = \mathsf{supp}(x) = \mathsf{supp}(u)$.
- *For $t = at'$ with $t' \in \bar{\mathbb{A}}^{n'} \times X, n = n' + 1$:* Then $u = au'$ with $t' \mathrel{\hat{\equiv}_\alpha} u'$ by Proposition 4.13. Thus, we get $a \in \mathsf{supp}(u)$. Furthermore, it follows from Lemma 4.11 and Lemma 4.10 that $\mathsf{FN}(t') = \mathsf{FN}(u') \subseteq \mathsf{supp}(u') \subseteq \mathsf{supp}(u)$.
- *For $t = \mathsf{l}at'$ with $t' \in \bar{\mathbb{A}}^{n'} \times X, n = n' + 1$:* By Proposition 4.13, we need to consider two cases for u:

  If $u = \mathsf{l}au'$ with $t' \mathrel{\hat{\equiv}_\alpha} u'$, then it follows from Lemma 4.11 and Lemma 4.10 that $\mathsf{FN}(t) \subseteq \mathsf{FN}(t') = \mathsf{FN}(u') \subseteq \mathsf{supp}(u') \subseteq \mathsf{supp}(u)$.

  If $u = \mathsf{l}bu'$ with $a \neq b$, $a \notin \mathsf{FN}(t')$, and $t' \mathrel{\hat{\equiv}_\alpha} (a\ b)u'$: Let $c \in \mathsf{FN}(t) = \mathsf{FN}(t') \setminus \{a\}$. If $c = b$, then $c \in \mathsf{supp}(\mathsf{l}bu')$. If $c \neq b$ (and by assumption $c \neq a$), it follows from Lemma 4.11 and Lemma 4.10 that $c \in (a\ b)\mathsf{FN}(t') = \mathsf{FN}((a\ b)t') = \mathsf{FN}(u') \subseteq \mathsf{supp}(u') \subseteq \mathsf{supp}(u)$.

□

We will also show some more properties useful when applying an equivariant function to a pretrace:

Lemma 4.15. *If $wx \in \bar{\mathbb{A}}^n \times X$ is a pretrace and $f : X \to Y$ is an equivariant function, then $\mathsf{FN}(w(f(x))) \subseteq \mathsf{FN}(wx)$.*

Proof. By induction on n.

- *For $w = \varepsilon$ with $n = 0$:* It follows from Proposition 2.6 that $\mathsf{FN}(f(x)) = \mathsf{supp}(f(x)) \subseteq \mathsf{supp}(x) = \mathsf{FN}(x)$.
- *For $w = aw'$ with $w' \in \bar{\mathbb{A}}^{n'}$, $n = n' + 1$:* By inductive hypothesis, we get $\mathsf{FN}(aw'(f(x))) = \mathsf{FN}(w'(f(x))) \cup \{a\} \subseteq \mathsf{FN}(w'x) \cup \{a\} = \mathsf{FN}(aw'x)$
- *For $w = \mathsf{l}aw'$ with $w' \in \bar{\mathbb{A}}^{n'}$, $n = n' + 1$:* By inductive hypothesis, we get $\mathsf{FN}(\mathsf{l}aw'(f(x))) = \mathsf{FN}(w'(f(x))) \setminus \{a\} \subseteq \mathsf{FN}(w'x) \setminus \{a\} = \mathsf{FN}(\mathsf{l}aw'x)$

□

Proposition 4.16. *Let $wx, vy \in \bar{\mathbb{A}}^n \times X$ be pretraces and $f : X \to Y$ an equivariant function. If $wx \mathrel{\hat{\equiv}_\alpha} vy$, then $w(f(x)) \mathrel{\hat{\equiv}_\alpha} v(f(y))$.*

Proof. By induction on n. We only need to consider the cases outlined by Proposition 4.13.

- *For $w = v = \varepsilon$ and $n = 0$:* Since $wx \mathrel{\hat{\equiv}_\alpha} vy$, we have $x = y$. It then follows that $f(x) = f(y)$ and, from Proposition 4.13 (1), $f(x) \mathrel{\hat{\equiv}_\alpha} f(y)$.
- *For $w = aw'$ and $v = av'$ with $w', v' \in \bar{\mathbb{A}}^{n'}$, $n = n' + 1$:* We know that $w'x \mathrel{\hat{\equiv}_\alpha} v'y$ and, by inductive hypothesis, $w'(f(x)) \mathrel{\hat{\equiv}_\alpha} v'(f(y))$. It then follows from Proposition 4.13 (2) that $aw'(f(x)) \mathrel{\hat{\equiv}_\alpha} av'(f(y))$.
- *For $w = \mathsf{l}aw'$ and $v = \mathsf{l}av'$ with $w', v' \in \bar{\mathbb{A}}^{n'}$, $n = n' + 1$:* Analogous to the above case.
- *For $w = \mathsf{l}aw'$ and $v = \mathsf{l}bv'$ with $a \neq b$, $w', v' \in \bar{\mathbb{A}}^{n'}$, $n = n' + 1$:* We know that $a \notin \mathsf{FN}(v'y)$ and $w'x \mathrel{\hat{\equiv}_\alpha} (a\ b)(v'y)$. It follows from Lemma 4.15 that $a \notin \mathsf{FN}(v'(f(y))) \subseteq \mathsf{FN}(v'y)$. Furthermore, we get $w'(f(x)) \mathrel{\hat{\equiv}_\alpha} (a\ b)(v'(f(y)))$ by inductive hypothesis. It then follows from Proposition 4.13 (4) that $\mathsf{l}aw'(f(x)) \mathrel{\hat{\equiv}_\alpha} \mathsf{l}bv'(f(y))$.

□

4.1.3 Injectivity for Local Freshness Semantics

Using this notion of alpha-equivalence between pretraces, we will now show that the local freshness semantics mapping $\hat{D}$ of pretraces is an injection from the equivalence classes modulo alpha-equivalence.

We will use the following properties for this:

Lemma 4.17. *Let $t, u \in \bar{\mathbb{A}}^{n+1} \times X$. If $\hat{D}([t]_{\hat{\alpha}}) = \hat{D}([u]_{\hat{\alpha}})$, then one of the following is true:*

1. *$t = at'$ and $u = au'$ for some $a \in \mathbb{A}$ and $t', u' \in \bar{\mathbb{A}}^n \times X$, or*
2. *$t = \shortmid at'$ and $u = \shortmid bu'$ for some $a, b \in \mathbb{A}$ and $t', u' \in \bar{\mathbb{A}}^n \times X$.*

Proof. We will consider the following exhaustive list of cases for t, u:

- *For $t = at'$ and $u = bu'$ with $t', u' \in \bar{\mathbb{A}}^{n'} \times X, n = n' + 1$:* If $a = b$, (1) follows directly. Suppose that $a \neq b$. Then $(a, \hat{\mathsf{ub}}(t')) = \hat{\mathsf{ub}}(at') \in \hat{D}([t]_{\hat{\alpha}})$. However, for all $\hat{\mathsf{ub}}(w) \in \hat{D}([u]_{\hat{\alpha}})$ with $w \in [u]_{\hat{\alpha}}$, we have $w \hat{\equiv}_\alpha u$. It follows from Proposition 4.13 that $w = bw'$ for some w' and thus $\hat{\mathsf{ub}}(w) = (b, \hat{\mathsf{ub}}(w')) \neq (a, \hat{\mathsf{ub}}(t'))$, contradicting the assumption.
- *For $t = \shortmid at'$ and $u = bu'$ with $t', u' \in \bar{\mathbb{A}}^{n'} \times X, n = n' + 1$:* Pick a $c \in \mathbb{A} \setminus \mathsf{supp}(t')$ with $c \neq a$ and $c \neq b$. By Proposition 4.13 (4), we know that $\shortmid at' \hat{\equiv}_\alpha \shortmid c((a\ c)t')$ and thus $(c, \hat{\mathsf{ub}}((a\ c)t')) \in \hat{D}([t]_{\hat{\alpha}})$. However, similarly to the above case, for all $\hat{\mathsf{ub}}(w) \in \hat{D}([u]_{\hat{\alpha}})$, we have $\hat{\mathsf{ub}}(w) = bw' \neq (c, \hat{\mathsf{ub}}((a\ c)t'))$ with some w', contradicting the assumption.
- *For $t = at'$ and $u = \shortmid bu'$ with $t', u' \in \bar{\mathbb{A}}^{n'} \times X, n = n' + 1$:* Analogous to the above case.
- *For $t = \shortmid at'$ and $u = \shortmid bu'$ with $t', u' \in \bar{\mathbb{A}}^{n'} \times X, n = n' + 1$:* (2) holds trivially.

□

Proposition 4.18. *Let $t' \in \bar{\mathbb{A}}^n \times X$, $a \in \mathbb{A}$, and $w' \in \mathbb{A}^n \times X$.*

1. *$bw' \in \hat{D}([at']_{\hat{\alpha}})$ iff $w' \in \hat{D}([t']_{\hat{\alpha}})$ and $a = b$.*
2. *$aw' \in \hat{D}([\shortmid at']_{\hat{\alpha}})$ iff $w' \in \hat{D}([t']_{\hat{\alpha}})$.*
3. *If $a \neq b$, then $bw' \in \hat{D}([\shortmid at']_{\hat{\alpha}})$ iff $b \notin \mathsf{FN}(t')$ and $w' \in \hat{D}([(a\ b)t']_{\hat{\alpha}})$.*

Proof. We will show each statement individually.

1. *'If':* Assume $w' \in \hat{D}([t']_{\hat{\alpha}})$ and $a = b$.

   Then there exists a $\tilde{w}' \in [t']_{\hat{\alpha}}$ with $\hat{\mathsf{ub}}(\tilde{w}') = w'$. Since $\tilde{w}' \hat{\equiv}_\alpha t'$, we get $a\tilde{w}' \hat{\equiv}_\alpha at'$ by Proposition 4.13 (2). Thus, it follows that

   $$aw' = (a, \hat{\mathsf{ub}}(\tilde{w}')) = \hat{\mathsf{ub}}(a\tilde{w}') \in \hat{D}([at']_{\hat{\alpha}}).$$

   'Only if': Assume $bw' \in \hat{D}([at']_{\hat{\alpha}})$.

   Then there exists a $\tilde{w} \in [at']_{\hat{\alpha}}$ with $\hat{\mathsf{ub}}(\tilde{w}) = bw'$. Since $\tilde{w} \hat{\equiv}_\alpha at'$, we know from Proposition 4.13 that $\tilde{w} = a\tilde{w}'$ for some $\tilde{w}' \in \bar{\mathbb{A}}^n \times X$ with $\tilde{w}' \hat{\equiv}_\alpha t'$. Thus, we have $\tilde{w}' \in [t']_{\hat{\alpha}}$. Furthermore, we get

   $$(a, \hat{\mathsf{ub}}(\tilde{w}')) = \hat{\mathsf{ub}}(a\tilde{w}') = \hat{\mathsf{ub}}(\tilde{w}) = bw',$$

   concluding that $a = b$ and $w' = \hat{\mathsf{ub}}(\tilde{w}') \in \hat{D}([t']_{\hat{\alpha}})$.

2. Analogous to the above proof.
3. *'If':* Assume $w' \in \hat{D}([(a\ b)t']_{\hat{\alpha}}$ and $b \notin \mathsf{FN}(t')$.

   Then there exists a $\tilde{w}' \in [(a\ b)t']_{\hat{\alpha}}$ with $\hat{\mathsf{ub}}(\tilde{w}') = w'$. Since $\tilde{w}' \mathrel{\hat{\equiv}_\alpha} (a\ b)t'$ and $b \notin \mathsf{FN}(t')$ with $a \neq b$, it follows from Proposition 4.13 (4) that $\mathsf{l}b\tilde{w}' \mathrel{\hat{\equiv}_\alpha} \mathsf{l}at'$. Thus, we get

   $$bw' = (b, \hat{\mathsf{ub}}(\tilde{w}')) = \hat{\mathsf{ub}}(\mathsf{l}b\tilde{w}') \in \hat{D}([\mathsf{l}at']_{\hat{\alpha}}).$$

   'Only if': Assume $bw' \in \hat{D}([\mathsf{l}at']_{\hat{\alpha}})$.

   Then there exists some $\tilde{w} \in [\mathsf{l}at']_{\hat{\alpha}}$ with $\hat{\mathsf{ub}}(\tilde{w}) = bw'$. Since $\tilde{w} \mathrel{\hat{\equiv}_\alpha} \mathsf{l}at'$, we know from Proposition 4.13 that $\tilde{w} = \mathsf{l}c\tilde{w}'$ for some $\tilde{w}' \in \bar{\mathbb{A}}^n \times X$ and $c \in \mathbb{A}$. Since

   $$(c, \hat{\mathsf{ub}}(\tilde{w}')) = \hat{\mathsf{ub}}(\mathsf{l}c\tilde{w}') = \hat{\mathsf{ub}}(\tilde{w}) = bw',$$

   we know that $c = b$ and $\hat{\mathsf{ub}}(\tilde{w}') = w'$. Since $b \neq a$ and $\mathsf{l}b\tilde{w}' \mathrel{\hat{\equiv}_\alpha} \mathsf{l}at'$, it follows from Proposition 4.13 that $b \notin \mathsf{FN}(t')$ and $\tilde{w}' \mathrel{\hat{\equiv}_\alpha} (a\ b)t'$. Finally, it follows that $w' = \hat{\mathsf{ub}}(\tilde{w}') \in \hat{D}([(a\ b)t']_{\hat{\alpha}})$.

□

Proposition 4.19. *The function $\hat{D}_n$ is injective for all $n \in \mathbb{N}_0$.*

Proof. Let $t, u \in \bar{\mathbb{A}}^n \times X$ with $\hat{D}([t]_{\hat{\alpha}}) = \hat{D}([u]_{\hat{\alpha}})$. We will show $[t]_{\hat{\alpha}} = [u]_{\hat{\alpha}}$ by induction on n.

- *For $t = x$ with $x \in X$:* Let $u = y \in X$. We know from Proposition 4.13 that $[x]_{\hat{\alpha}} = \{x\}$ and $[y]_{\hat{\alpha}} = \{y\}$. Thus, we have

  $$\hat{D}([t]_{\hat{\alpha}}) = \{\hat{\mathsf{ub}}(x)\} = \{x\} = \{y\} = \{\hat{\mathsf{ub}}(y)\} = \hat{D}([u]_{\hat{\alpha}}).$$

  This means that $x = y$ and, by Proposition 4.13 (1), we get $t = x \mathrel{\hat{\equiv}_\alpha} y = u$.

- *For $t = at'$ with $t' \in \bar{\mathbb{A}}^{n'} \times X, n = n' + 1$:* We know from Lemma 4.17 that $u = au'$ for some $u' \in \bar{\mathbb{A}}^{n'} \times X$. We will show that $\hat{D}([t']_{\hat{\alpha}}) = \hat{D}([u']_{\hat{\alpha}})$:

  $$\begin{aligned} w' \in \hat{D}([t']_{\hat{\alpha}}) \quad &\Leftrightarrow \quad aw' \in \hat{D}([at']_{\hat{\alpha}}) && \text{by Proposition 4.18 (1)} \\ &\Leftrightarrow \quad aw' \in \hat{D}([au']_{\hat{\alpha}}) && \text{because } \hat{D}([t]_{\hat{\alpha}}) = \hat{D}([u]_{\hat{\alpha}}) \\ &\Leftrightarrow \quad w' \in \hat{D}([u']_{\hat{\alpha}}) && \text{by Proposition 4.18 (1).} \end{aligned}$$

  It now follows by inductive hypothesis that $t' \mathrel{\hat{\equiv}_\alpha} u'$ and, by Proposition 4.13 (2), that $t = at' \mathrel{\hat{\equiv}_\alpha} au' = u$.

- *For $t = \mathsf{l}at'$ with $t' \in \bar{\mathbb{A}}^{n'} \times X, n = n' + 1$:* We know from Lemma 4.17 that $u = \mathsf{l}bu'$ for some $u' \in \bar{\mathbb{A}}^{n'} \times X$ and $b \in \mathbb{A}$.

  If $a = b$: One can show that $\hat{D}([t']_{\hat{\alpha}}) = \hat{D}([u']_{\hat{\alpha}})$ analogously to the above case. It then follows by inductive hypothesis that $t' \mathrel{\hat{\equiv}_\alpha} u'$ and, by Proposition 4.13 (3), we get $t = \mathsf{l}at' \mathrel{\hat{\equiv}_\alpha} \mathsf{l}au' = u$.

  If $a \neq b$: First note that $(a, \hat{\mathsf{ub}}(t')) = \hat{\mathsf{ub}}(t) \in \hat{D}([t]_{\hat{\alpha}}) = \hat{D}([\mathsf{l}bu']_{\hat{\alpha}})$. Thus, it follows from Proposition 4.18 (3) that $a \notin \mathsf{FN}(u')$.

  We will now show that $\hat{D}([t']_{\hat{\alpha}}) = \hat{D}([(a\ b)u']_{\hat{\alpha}})$. We have

  $$\begin{aligned} w' \in \hat{D}([t']_{\hat{\alpha}}) \quad &\Longrightarrow \quad aw' \in \hat{D}([\mathsf{l}at']_{\hat{\alpha}}) && \text{by Proposition 4.18 (2)} \\ &\Longrightarrow \quad aw' \in \hat{D}([\mathsf{l}bu']_{\hat{\alpha}}) && \text{because } \hat{D}([t]_{\hat{\alpha}}) = \hat{D}([u]_{\hat{\alpha}}) \\ &\Longrightarrow \quad w' \in \hat{D}([(a\ b)u']_{\hat{\alpha}}) && \text{by Proposition 4.18 (3),} \end{aligned}$$

and conversely

$$\begin{aligned}
w' \in \hat{D}([(a\ b)u']_{\hat{\alpha}}) &\implies (a\ b)w' \in \hat{D}([u']_{\hat{\alpha}}) && \text{by Proposition 2.12, Lemma 4.7}\\
&\implies (b, (a\ b)w') \in \hat{D}([|bu']_{\hat{\alpha}}) && \text{by Proposition 4.18 (2)}\\
&\implies (b, (a\ b)w') \in \hat{D}([|at']_{\hat{\alpha}}) && \text{because } \hat{D}([t]_{\hat{\alpha}}) = \hat{D}([u]_{\hat{\alpha}})\\
&\implies (a\ b)w' \in \hat{D}([(a\ b)t']_{\hat{\alpha}}) && \text{by Proposition 4.18 (3)}\\
&\implies w' \in \hat{D}([t']_{\hat{\alpha}}) && \text{by Proposition 2.12, Lemma 4.7.}
\end{aligned}$$

It now follows by inductive hypothesis that $t' \hat{\equiv}_\alpha (a\ b)u'$. Finally, since since $a \notin \mathsf{FN}(u')$, we get $t = |at' \hat{\equiv}_\alpha |bu' = u$ from Proposition 4.13 (4).

□

4.2 Local Freshness Semantics as a Model

In the following section, we will fix a nominal set X, and a depth $n \leq \omega$.

We will now define a model which interprets interprets pretraces as defined in Definition 4.6 and extends to all pretrace terms. Defining this as a model is useful because we can benefit from the soundness already proven before. In the next section, we will show that this interpretation is indeed injective for all pretrace terms modulo derivable equality.

Definition 4.20. The (Σ, n)-algebra $F'(X)$ is defined as follows:

- $(F'(X))_m := \mathcal{P}_{\mathsf{fs}}(\mathbb{A}^m \times X)$,
- $\mathsf{pre}_{F'(X),m}(a, L) := \{aw : w \in L\}$,
- $\mathsf{abs}_{F'(X),m}(\langle a\rangle L) := \{bv : \langle a\rangle L = \langle a'\rangle L', w' \in L', \langle a'\rangle w' = \langle b\rangle v\}$,
- $+_{F'(X),m}(L_1, L_2) := L_1 \cup L_2$,
- $\perp_{F'(X),m} := \emptyset$.

Lemma 4.21. *The above description of $F'(X)$ is indeed a well-defined nominal (Σ, n)-algebra, in that*

1. $\mathsf{pre}_{F'(X),m}$ *is equivariant for all m.*
2. $\mathsf{abs}_{F'(X),m}$ *is equivariant for all m.*

Proof. We will prove each statement individually: Let $a \in \mathbb{A}, w \in (F'(X))_m$ and $\pi \in \mathsf{Perm}(\mathbb{A})$.

1. Then

$$\mathsf{pre}_{F'(X),m}(\pi a, \pi L) = \{(\pi a, \pi w) : \pi w \in \pi L\} = \{(\pi a, \pi w) : w \in L\} = \pi \mathsf{pre}_{F'(X),m}(a, L).$$

It follows that $\mathsf{pre}_{F'(X),m}(a, L)$ is finitely supported by $\mathsf{supp}(L) \cup \{a\}$.

2. Then

$$\begin{aligned}
&\mathsf{abs}_{F'(X),m}(\pi\langle a\rangle L) \\
&= \{(\pi b, \pi v) : \pi\langle a\rangle L = \langle \pi a'\rangle \pi L', \pi w' \in \pi L', \langle \pi a'\rangle \pi w' = \langle \pi b\rangle \pi v\} && \text{by definition} \\
&= \{(\pi b, \pi v) : \langle a\rangle L = \langle a'\rangle L', \pi w' \in \pi L', \langle \pi a'\rangle \pi w' = \langle \pi b\rangle \pi v\} && \text{by Proposition 2.9} \\
&= \{(\pi b, \pi v) : \langle a\rangle L = \langle a'\rangle L', w' \in L', \langle \pi a'\rangle \pi w' = \langle \pi b\rangle \pi v\} && \text{by definition} \\
&= \{(\pi b, \pi v) : \langle a\rangle L = \langle a'\rangle L', w' \in L', \langle a'\rangle w' = \langle b\rangle v\} && \text{by Proposition 2.9} \\
&= \pi \mathsf{abs}_{F'(X),m}(\pi\langle a\rangle L).
\end{aligned}$$

It follows that $\mathsf{abs}_{F'(X),m}(\langle a\rangle L)$ is finitely supported by $\mathsf{supp}(\langle a\rangle L)$.

□

The slightly convoluted definition for $\mathsf{abs}_{F'(X),m}$ is needed to ensure that it is well-defined and that the algebra satisfies the axioms of the theory:

Example 4.22. If we do not quantify over all representatives of $\langle a\rangle L$, then the definition would not be well-defined:

Consider $X = \mathbb{A} = \{a, b, c, \ldots\}$ and $L = \{ax : x \in \mathbb{A}\} \in (F'(\mathbb{A}))_1$.

Let $h_m : \mathbb{A} \times (F'(X))_m$ be defined by $h_m(a, L) = \{bv : w \in L, \langle a\rangle w = \langle b\rangle v\}$. Then $aab \in h_1(a, L)$.

Let $L' = \{bx : x \in \mathbb{A}\}$. Then $\langle a\rangle L = \langle b\rangle L'$. However, $aab \notin h_1(b, L')$: Otherwise, there would be some $bx \in L'$ with $\langle b\rangle bx = \langle a\rangle ab$, but $b \in \mathsf{supp}(ab)$.

Example 4.23. If we do not quantify over all representatives of $\langle a'\rangle w'$, then the algebra would not satisfy the axioms:

Consider $X = 1 = \{\star\}$ and $\mathbb{A} = \{a, b, c, \ldots\}$.

Let A be the (Σ, n)-algebra defined just like Definition 4.20 but with

$$\mathsf{abs}_{A,m}(\langle a\rangle L) = \{a'w' : \langle a\rangle L = \langle a'\rangle L', w' \in L'\}.$$

Consider $t = \mathsf{l}a(a\star + b\star)$ and $u = \mathsf{l}aa\star + \mathsf{l}ab\star$. Then $X \vdash_2 t = u$ is derivable by $(\mathrm{ax}_{r=s})$ and axiom (4.6).

Let $\iota : 1 \to A_0$ be the equivariant environment with $\iota(\star) = \{\star\}$. Then

$$[\![t]\!]_2^\iota = \mathsf{abs}_{A,1}(\langle a\rangle\{a\star, b\star\}) = \{aa\star, ab\star, cc\star, cb\star, dd\star, db\star, \ldots\}$$

and

$$[\![u]\!]_2^\iota = \mathsf{abs}_{A,1}(\langle a\rangle\{a\star\}) \cup \mathsf{abs}_{A,1}(\langle a\rangle\{b\star\}) = \{aa\star, bb\star, cc\star, \ldots\} \cup \{ab\star, cb\star, db\star, \ldots\},$$

are not equal.

We will now show that, with this extended definition, the algebra indeed satisfies all of the axioms of the theory. Intuitively, a problem arises when individual summands have smaller or larger support than that of the entire term. Because of this, $\langle a\rangle L = \langle a'\rangle L'$ for the interpretation L of the sum term does not always imply $\langle a\rangle S = \langle a'\rangle S'$ for the interpretation S of a summand term and $S' = (a\ a')S$ and vice versa. With this definition, if $w' \in S'$, we can find a $w'' \in S''$ with $\langle a\rangle S = \langle c\rangle S''$ and $\langle c\rangle w'' = \langle a'\rangle w'$. To show this, we will use the following lemma:

Lemma 4.24. *If $a, a', c \in \mathbb{A}$ are pairwise distinct, then $(a'\ c) = (a\ c)(a'\ c)(a\ a')$.*

Proof. By simple computation. □

Lemma 4.25 (Monotonicity of abs)**.** *If $L \subseteq S$, then $\mathsf{abs}_{F'(X),m}(\langle a \rangle L) \subseteq \mathsf{abs}_{F'(X),m}(\langle a \rangle S)$.*

Proof. Let $bv \in \mathsf{abs}_{F'(X),m}(\langle a \rangle L)$ with $\langle a \rangle L = \langle a' \rangle L'$, $w' \in L'$, and $\langle a' \rangle w' = \langle b \rangle v$. By Proposition 2.8, we again need to consider two cases:

If $a = a'$ and $L' = L$, then $\langle a \rangle S = \langle a' \rangle S$ with $w' \in L' \subseteq S$, and thus $bv \in \mathsf{abs}_{F'(X),m}(\langle a \rangle S)$.

If $a \neq a'$ then $a' \# L$ and $L = (a\ a')L'$. We pick some $c \in \mathbb{A} \setminus \mathsf{supp}(a, a', b, L, S, w')$. We then have

$$\begin{aligned}
w' \in L' \quad &\Longrightarrow \quad (a\ a')w' \in L && \text{because } L = (a\ a')L' \\
&\Longrightarrow \quad (a'\ c)(a\ a')w' \in L && \text{because } (a'\ c)L = L \\
&\Longrightarrow \quad (a'\ c)(a\ a')w' \in S && \text{because } L \subseteq S \\
&\Longrightarrow \quad (a\ c)(a'\ c)(a\ a')w' \in (a\ c)S && \text{by definition} \\
&\Longrightarrow \quad (a'\ c)w' \in (a\ c)S && \text{by Lemma 4.24.}
\end{aligned}$$

Because we have $a \neq c$ and $c \# S$, it follows from Proposition 2.8 that $\langle a \rangle S = \langle c \rangle (a\ c)S$.

Furthermore, since $c \neq a'$ and $c \# w'$, it follows from Proposition 2.8 that $\langle c \rangle (a'\ c)w' = \langle a' \rangle w' = \langle b \rangle v$, and thus $bv \in \mathsf{abs}_{F'(X),m}(\langle a \rangle S)$. □

Proposition 4.26. *The definition of $F'(X)$ given in Definition 4.20 is a (T, n)-model.*

Proof. Since we know that $F'(X)$ is a nominal (Σ, n)-algebra (by Lemma 4.21), we only need to show that $F'(X)$ satisfies every axiom.

Let Y be a nominal set with $x, y, z \in Y$ and $\iota : Y \to (F'(X))_k$ an equivariant environment.

- *For Equation 4.1:* By commutativity of $\cup$, we have $[\![x + y]\!]^\iota_0 = \iota(x) \cup \iota(y) = \iota(y) \cup \iota(x) = [\![y + x]\!]^\iota_0$.
- *For Equation 4.2:* By associativity of $\cup$, we have $[\![(x + y) + z]\!]^\iota_0 = (\iota(x) \cup \iota(y)) \cup \iota(z) = \iota(x) \cup (\iota(y) \cup \iota(z)) = [\![x + (y + z)]\!]^\iota_0$.
- *For Equation 4.3:* By idempotence of $\cup$, we have $[\![x + x]\!]^\iota_0 = \iota(x) \cup \iota(x) = \iota(x) = [\![x]\!]^\iota_0$.
- *For Equation 4.4:* By definition, we have $[\![x + \bot]\!]^\iota_0 = \iota(x) \cup \emptyset = [\![x]\!]^\iota_0$.
- *For Equation 4.5:* By definition, we have

$$\begin{aligned}
[\![a(x + y)]\!]^\iota_1 &= \mathsf{pre}_{F'(X),k}(a, \iota(x) \cup \iota(y)) \\
&= \{(a, w) : w \in \iota(x) \cup \iota(y)\} \\
&= \{(a, w) : w \in \iota(x)\} \cup \{(a, w) : w \in \iota(y)\} \\
&= \mathsf{pre}_{F'(X),k}(\iota(x)) \cup \mathsf{pre}_{F'(X),k}(\iota(y)) \\
&= [\![ax + ay]\!]^\iota_1.
\end{aligned}$$

- *For Equation 4.6:* We have to show that $[\![\mathsf{l}a(x + y)]\!]^\iota_1 = \mathsf{abs}_{F'(X),k}(\langle a \rangle (\iota(x) \cup \iota(y))) = \mathsf{abs}_{F'(X),k}(\langle a \rangle \iota(x)) \cup \mathsf{abs}_{F'(X),k}(\langle a \rangle \iota(y)) = [\![\mathsf{l}ax + \mathsf{l}by]\!]^\iota_1$.

  First, suppose $bv \in \mathsf{abs}_{F'(X),k}(\langle a \rangle (\iota(x) \cup \iota(y)))$ with $\langle a \rangle (\iota(x) \cup \iota(y)) = \langle a' \rangle L'$, $w' \in L'$, and $\langle a' \rangle w' = \langle b \rangle v$. We consider the two cases outlined in Proposition 2.8:

  If $a = a'$ and $L' = \iota(x) \cup \iota(y)$, we know that w.l.o.g. $w' \in \iota(x)$. It then immediately follows that $\langle a \rangle \iota(x) = \langle a' \rangle \iota(x)$, and thus $bv \in \mathsf{abs}_{F'(X),m}(\langle a \rangle \iota(x))$.

If $a \neq a'$ then $a' \# (\iota(x) \cup \iota(y))$ and $\iota(x) \cup \iota(y) = (a\ a')L'$. We pick some $c \in \mathbb{A} \setminus \mathsf{supp}(a, a', b, \iota(x), \iota(y), w')$. It then follows that

$$\begin{aligned} w' \in L' &\implies (a\ a')w' \in \iota(x) \cup \iota(y) && \text{because } \iota(x) \cup \iota(y) = (a\ a')L' \\ &\implies (a'\ c)(a\ a')w' \in \iota(x) \cup \iota(y) && \text{because } (a'\ c)(\iota(x) \cup \iota(y)) = \iota(x) \cup \iota(y). \end{aligned}$$

We assume w.l.o.g. that $(a'\ c)(a\ a')w' \in \iota(x)$. It then follows from Lemma 4.24 that $(a'\ c)w' = (a\ c)(a'\ c)(a\ a')w' \in (a\ c)\iota(x)$.

Because we have $a \neq c$ and $c \# \iota(x)$, it follows from Proposition 2.8 that $\langle a \rangle \iota(x) = \langle c \rangle (a\ c)\iota(x)$. Furthermore, since $c \neq a'$ and $c \# w'$, it follows from Proposition 2.8 that $\langle c \rangle (a'\ c)w' = \langle a' \rangle w' = \langle b \rangle v$, and thus $bv \in \mathsf{abs}_{F'(X),m}(\langle a \rangle \iota(x))$.

Conversely, suppose w.l.o.g. that $bv \in \mathsf{abs}_{F'(X),k}(\langle a \rangle \iota(x))$. Since $\iota(x) \subseteq \iota(x) \cup \iota(y)$, $bv \in \mathsf{abs}_{F'(X),k}(\langle a \rangle (\iota(x) \cup \iota(y)))$ follows directly from Lemma 4.25.

- *For Equation 4.7:* By definition, we have

$$\begin{aligned} [\![a\bot]\!]_1^\iota &= \mathsf{pre}_{F'(X),0}(a, \emptyset) \\ &= \{(a, w) : w \in \emptyset\} \\ &= \emptyset = [\![\bot]\!]_1^\iota. \end{aligned}$$

- *For Equation 4.8:* Since $\langle a \rangle \emptyset = \langle a' \rangle L'$ implies $L' = (a\ a')\emptyset = \emptyset$, we have

$$\begin{aligned} [\![\mathsf{l}a\bot]\!]_1^\iota &= \mathsf{abs}_{F'(X),0}(\langle a \rangle \emptyset) \\ &= \{bv : \langle a \rangle \emptyset = \langle a' \rangle L', w' \in L', \langle a' \rangle w' = \langle b \rangle v\} \\ &= \{bv : \langle a \rangle \emptyset = \langle a' \rangle \emptyset, w' \in \emptyset, \langle a' \rangle w' = \langle b \rangle v\} \\ &= \emptyset = [\![\bot]\!]_1^\iota. \end{aligned}$$

- *For Equation 4.9:* We will show $[\![ax]\!]_1^\iota = \mathsf{pre}_{F'(X),k}(a, \iota(x)) \subseteq \mathsf{abs}_{F'(X),k}(\langle a \rangle \iota(x)) = [\![\mathsf{l}ax]\!]_1^\iota$, which is enough to show that $[\![\mathsf{l}ax]\!]_1^\iota = [\![\mathsf{l}ax]\!]_1^\iota \cup [\![ax]\!]_1^\iota = [\![\mathsf{l}ax + ax]\!]_1^\iota$.

  Let $aw \in \mathsf{pre}_{F'(X),k}(a, \iota(x))$ with $w \in \iota(x)$. Since $\langle a \rangle \iota(x) = \langle a \rangle \iota(x)$ and $\langle a \rangle w = \langle a \rangle w$ are trivially satisfied, we have $aw \in \mathsf{abs}_{F'(X),k}(\langle a \rangle \iota(x))$.

□

Note that, while the naming might hint at this definition of $F'(X)$ directly inducing a functor $F' : \mathsf{Nom} \to \mathsf{Alg}(T, n)$, this is not the case:

Example 4.27. Consider the nominal sets $X = \mathbb{A} = \{a, b, \ldots\}$ and $Y = 1 = \{\star\}$ with $! : \mathbb{A} \to 1$ being the morphism into the terminal object.

Let $h_m : \mathcal{P}_{\mathsf{fs}}(A^m \times \mathbb{A}) \to \mathcal{P}_{\mathsf{fs}}(\mathbb{A}^m \times 1)$ be the mapping of $!$ using the functor $\mathcal{P}_{\mathsf{fs}}(\mathbb{A}^m \times -)$ on Nom, namely

$$h_m(L) = \{w\star : w \in \mathbb{A}^m, wx \in L\}.$$

We will show that $(h_m)_{0 \leq m \leq n}$ does not define a morphism between (Σ, n)-algebras $F'(\mathbb{A}) \to F'(1)$.

Consider $L = \{a\} \in (F'(\mathbb{A}))_0$. Then Definition 3.12 (3) would imply that $h_1(\mathsf{abs}_{F'(X),0}(\langle b \rangle L)) = \mathsf{abs}_{F'(X),0}(\langle b \rangle h_0(L))$, however

$$h_1(\mathsf{abs}_{F'(X),0}(\langle b \rangle L)) = h_1(\{b'a' : \langle b \rangle a = \langle b' \rangle a'\}) = h_1(\{ba, ca, \ldots\}) = \{b\star, c\star, \ldots\}$$

and

$$\mathsf{abs}_{F'(X),0}(\langle b\rangle h_0(L)) = \mathsf{abs}_{F'(X),0}(\langle b\rangle\{\star\}) = \{b'\star : \langle b\rangle\star = \langle b'\rangle\star\} = \{a\star, b\star, c\star, \ldots\}$$

are not equal.

However, this definition is enough to define an interpretation of pretrace terms into this model:

Definition 4.28. Let the morphism $\Phi : F(X) \to F'(X)$ be defined as

$$\Phi_m([t]_m) = [\![t]\!]_m^{\eta'_X},$$

where the equivariant environment $\eta'_X : X \to (F'(X))_0$ is defined as

$$\eta'_X(x) = \{x\}.$$

Lemma 4.29. *The above description of Φ is indeed a well-defined morphism between (Σ, n)-algebras.*

Proof. We first note that Φ is well-defined because the derivation system is sound (as shown in Theorem 3.17) and $F'(X)$ is a (T, n)-model (as shown in Proposition 4.26).

To show the morphism property in Definition 3.12 for pre, let $[t]_m \in (F(X))_m$ and $a \in \mathbb{A}$. It immediately follows that

$$\begin{aligned}
\Phi_{m+1}(\mathsf{pre}_{F(X),m}(a, [t]_m)) &= \Phi_{m+1}([at]_{m+1}) && \text{by Definition 3.19}\\
&= [\![at]\!]_{m+1}^{\eta'_X} && \text{by Definition 4.28}\\
&= \mathsf{pre}_{F'(X),m}(a, [\![t]\!]_m^{\eta'_X}) && \text{by Definition 3.13}\\
&= \mathsf{pre}_{F'(X),m}(a, \Phi_m([t]_m)) && \text{by Definition 4.28.}
\end{aligned}$$

For the other operations, the steps are analogous to the above. □

Furthermore, we will show that, in the context of single pretraces, the interpretation in this model is exactly the local freshness semantics of the pretrace.

Theorem 4.30. *If $t \in \bar{\mathbb{A}}^m \times X$ is a pretrace, then $\Phi_m([t]_m) = \hat{D}([t]_{\hat{\alpha}})$.*

Proof. By induction on m.

- *For $t = x$ with $x \in X$:* Since $[x]_{\hat{\alpha}} = \{x\}$ by Proposition 4.13, we get

$$\Phi_0([x]_0) = \eta'_X(x) = \{x\} = \{\hat{\mathsf{ub}}(x)\} = \hat{D}([x]_{\hat{\alpha}}).$$

- *For $t = at'$ with $t' \in \bar{\mathbb{A}}^{m'} \times X, m = m' + 1$:* We have

$$\begin{aligned}
\Phi_m([at']_m) &= \mathsf{pre}_{F'(X),m'}(a, \Phi_{m'}([t']_{m'})) && \text{by Definition 3.13}\\
&= \{aw : w \in \Phi_{m'}([t']_{m'})\} && \text{by Definition 4.20}\\
&= \{aw : w \in \hat{D}([t']_{\hat{\alpha}})\} && \text{by inductive hypothesis.}
\end{aligned}$$

  First, assume $aw \in \Phi_m([at']_m)$ with $w \in \hat{D}([t']_{\hat{\alpha}})$. It follows from Proposition 4.18 (1) that $aw \in \hat{D}([at']_{\hat{\alpha}})$.

  Conversely, assume $bw' \in \hat{D}([at']_{\hat{\alpha}})$. It follows from Proposition 4.18 (1) that $a = b$ and $w' \in \hat{D}([t']_{\hat{\alpha}})$. Thus, we get $bw' = aw' \in \Phi_m([at']_m)$ by definition.

- *For $t = \mathsf{l}at'$ with $t' \in \bar{\mathbb{A}}^{m'} \times X, m = m' + 1$:* We have

$$
\begin{aligned}
&\Phi_m([\mathsf{l}at']_m) \\
&= \mathsf{abs}_{F'(X),m'}(\langle a \rangle \Phi_{m'}([t']_{m'})) && \text{by Definition 3.13} \\
&= \{bv : \langle a \rangle \Phi_{m'}([t']_{m'}) = \langle a' \rangle L', w' \in L', \langle a' \rangle w' = \langle b \rangle v\} && \text{by Definition 4.20} \\
&= \{bv : \langle a \rangle \hat{D}([t']_{\hat{\alpha}}) = \langle a' \rangle L', w' \in L', \langle a' \rangle w' = \langle b \rangle v\} && \text{by inductive hypothesis.}
\end{aligned}
$$

  Assume $bv \in \Phi_m([\mathsf{l}at']_m)$ with $\langle a \rangle \hat{D}([t']_{\hat{\alpha}}) = \langle a' \rangle L'$, $w' \in L'$, and $\langle a' \rangle w' = \langle b \rangle v$.

  We'll first show that $\mathsf{l}a'\tilde{w}' \in [t]_{\hat{\alpha}}$ for some $\tilde{w}' \in \mathbb{A}^{m'} \times X$ with $w' = \hat{\mathsf{ub}}(\tilde{w}')$:

  - If $a = a'$, then $L' = \hat{D}([t']_{\hat{\alpha}})$ and $w' \in \hat{D}([t']_{\hat{\alpha}})$. By definition, there exists a $\tilde{w}' \in [t']_{\hat{\alpha}}$ with $w' = \hat{\mathsf{ub}}(\tilde{w}')$. It then follows from Proposition 4.13 (3) that $\mathsf{l}a\tilde{w}' \in [at']_{\hat{\alpha}}$.
  - If $a \neq a'$, then $a' \notin \mathsf{supp}(\hat{D}([t']_{\hat{\alpha}})) = \mathsf{supp}([t']_{\hat{\alpha}}) = \mathsf{FN}(t')$ by Proposition 4.19 and Proposition 4.14. Furthermore, we have $w' \in (a\ a')\hat{D}([t']_{\hat{\alpha}}) = \hat{D}([(a\ a')t']_{\hat{\alpha}}$. By definition, there exists a $\tilde{w}' \in [(a\ a')t']_{\hat{\alpha}}$ with $w' = \hat{\mathsf{ub}}(\tilde{w}')$. It follows from Proposition 4.13 (4) that $\mathsf{l}a'\tilde{w}' \in [at']_{\hat{\alpha}}$.

  We'll proceed to show that $bv \in \hat{D}([t]_{\hat{\alpha}})$:

  - If $a' = b$, then $v = w'$ and $bv = (a', \hat{\mathsf{ub}}(\tilde{w}')) = \hat{\mathsf{ub}}(\mathsf{l}a'\tilde{w}') \in \hat{D}([t]_{\hat{\alpha}})$.
  - If $a' \neq b$, then $b\#w'$ and $v = (a'\ b)w'$. Then we have $b \notin \mathsf{FN}(w') \subseteq \mathsf{supp}(w')$. It follows from Proposition 4.13 (4) that $\mathsf{l}b((a'\ b)\tilde{w}') \in [t]_{\hat{\alpha}}$. Thus, we get

$$bv = (b, (a'\ b)w') = (b, (a'\ b)\hat{\mathsf{ub}}(\tilde{w}')) = (b, \hat{\mathsf{ub}}((a'\ b)\tilde{w}')) = \hat{\mathsf{ub}}(\mathsf{l}b((a'\ b)\tilde{w}')) \in \hat{D}([t]_{\hat{\alpha}}).$$

  Conversely, assume that $\hat{\mathsf{ub}}(\tilde{w}) \in \hat{D}([\mathsf{l}at']_{\hat{\alpha}})$ with $\tilde{w} \in [\mathsf{l}at']_{\hat{\alpha}}$. It follows from Proposition 4.13 that we only need to consider two cases:

  - If $\tilde{w} = \mathsf{l}a\tilde{w}'$ with $\tilde{w}' \hat{\equiv}_\alpha t'$: We know that $\hat{\mathsf{ub}}(\tilde{w}') \in \hat{D}([t']_{\hat{\alpha}})$. With $a' = b = a$ and $L' = \hat{D}([t']_{\hat{\alpha}})$, it then follows that $\hat{\mathsf{ub}}(\tilde{w}) = (a, \hat{\mathsf{ub}}(\tilde{w}')) \in \Phi_m([\mathsf{l}at']_m)$.
  - If $\tilde{w} = \mathsf{l}a'\tilde{w}'$ with $a \neq a'$, $a' \notin \mathsf{FN}(t')$, and $\tilde{w}' \hat{\equiv}_\alpha (a\ a')t'$: We know that $a' \notin \mathsf{supp}(\hat{D}([t']_{\hat{\alpha}})) = \mathsf{supp}([t']_{\hat{\alpha}}) = \mathsf{FN}(t')$ by Proposition 4.19 and Proposition 4.14. It follows from Proposition 2.8 that

$$\langle a \rangle \hat{D}([t']_{\hat{\alpha}}) = \langle a' \rangle (a\ a')\hat{D}([t']_{\hat{\alpha}}) = \langle a' \rangle \hat{D}([(a\ a')t']_{\hat{\alpha}}).$$

    Since $\tilde{w}' \in [(a\ a')t']_{\hat{\alpha}}$ and thus $\hat{\mathsf{ub}}(\tilde{w}') \in \hat{D}([(a\ a')t']_{\hat{\alpha}})$, by setting $b = a'$, we get $\hat{\mathsf{ub}}(\tilde{w}) = (a', \hat{\mathsf{ub}}(\tilde{w}')) \in \Phi_m([\mathsf{l}at']_m)$.

□

This characterization will enable us to prove the injectivity of Φ_m: In particular, it tells us that when applying abs to a pretrace, one only has to consider the "outer" abstraction in the definition, since $\mathsf{supp}(L') = \mathsf{FN}(w') \subseteq \mathsf{supp}(w')$ for all $w' \in L'$.

4.3 Interpretation of Sums

Having shown that Φ extends our definition of local freshness semantics to pretrace terms with $+$ and $\bot$, we will proceed to give a more explicit description of this interpretation.

Before we continue, we will prove some helpful properties of more complex terms.

Lemma 4.31. *Let $t, u, v \in \mathsf{Term}_{\Sigma,m}(X)$ and $a \in \mathbb{A}$. Then the following equalities hold:*

1. $[t + u]_m = [u + t]_m$,
2. $[(t + u) + v]_m = [t + (u + v)]_m$,
3. $[t + t]_m = [t]_m$,
4. $[t + \bot]_m = [t]_m$,
5. $[a(t + u)]_m = [at + au]_m$,
6. $[\mathsf{I}a(t + u)]_m = [\mathsf{I}at + \mathsf{I}u]_m$,
7. $[a\bot]_{m+1} = [\bot]_{m+1}$,
8. $[\mathsf{I}a\bot]_{m+1} = [\bot]_{m+1}$,
9. $[\mathsf{I}at]_{m+1} = [\mathsf{I}at + at]_{m+1}$.

Proof. By simple application of $(\mathrm{ax}_{r=s})$, for example for (1):

We know that $\mathsf{Term}_{\Sigma,m}(X) \vdash_0 t + u = u + t$ is an axiom in the graded theory by Definition 4.1. Set $\tau = \mathsf{id}_{\mathbb{A}} \in \mathsf{Perm}(\mathbb{A})$ and $\sigma = id_{\mathsf{Term}_{\Sigma,m}(X)} : \mathsf{Term}_{\Sigma,m}(X) \to \mathsf{Term}_{\Sigma,m}(X)$. Then σ is trivially derivably equivariant: Let $\pi \in \mathsf{Perm}(\mathbb{A})$ and $r \in \mathsf{Term}_{\Sigma,m}(X)$. Then $X \vdash_m \pi\sigma(r) = \sigma(\pi r)$ is derivable because $\pi\sigma(r) = \pi r = \sigma(\pi r)$ and because derivable equality is an equivalence relation by Lemma 3.18.

Thus, we can conclude that $X \vdash_{m+0} (\tau(t + u))\sigma = (\tau(u + t))\sigma$ is derivable by applying $(\mathrm{ax}_{t+u=u+t})$. Since $(\tau(t + u))\sigma = (t + u)\sigma = t + u$ and $(\tau(u + t))\sigma = (u + t)\sigma = u + t$, we also get $X \vdash_m t + u = u + t$. By definition of $\sim$, this means $[t + u]_m = [u + t]_m$. ☐

Notation 4.32. Since summation in equivalence classes is both commutative and associative (see Lemma 4.31 (1) and (2)), we will leave out the parenthesis, implicitly add or remove parentheses, and reorder summands within equivalence classes. The constant $\bot$ is seen as a neutral element for summation (by Lemma 4.31 (4)).

We will also often implicitly use that derivable equality is a congruence (by Lemma 3.18).

Furthermore, we will use the notation

$$\left[\sum_{i=1}^{k} t_i\right]_m \coloneqq [t_1 + \ldots + t_k]_m,$$

where $k \in \mathbb{N}_0$ and $t_1, \ldots, t_k \in \mathsf{Term}_{\Sigma,m}(X)$. For $k = 0$, we set

$$\left[\sum_{i=1}^{0} t_i\right]_m \coloneqq \bot.$$

If $S = \{t_1, \ldots, t_k\} \subseteq \mathsf{Term}_{\Sigma,m}(X)$ is a finite set with $k \in \mathbb{N}_0$, we also use the notation

$$\left[\sum_{t\in S} t\right]_m \coloneqq \left[\sum_{i=1}^{k} t_i\right].$$

Since, by Lemma 4.31 (3), sums are idempotent, it does not matter whether $t_1, \ldots, t_k$ are pairwise distinct or not.

Lemma 4.33 (Distributivity over Sums)**.** *Let $k \in \mathbb{N}_0$ and $t_1, \ldots, t_k \in \mathsf{Term}_{m,\Sigma}(X)$ be terms. Then the following equalities hold:*

1. $\left[a\left(\sum_{i=1}^{k} t_i\right)\right]_{m+1} = \left[\sum_{i=1}^{k} at_i\right]_{m+1}$ *for all $a \in \mathbb{A}$,*

2. $\left[\mathsf{la}\left(\sum_{i=1}^{k} t_i\right)\right]_{m+1} = \left[\sum_{i=1}^{k} \mathsf{la} t_i\right]_{m+1}$ *for all* $a \in \mathbb{A}$,
3. $\left[\pi\left(\sum_{i=1}^{k} t_i\right)\right]_{m} = \left[\sum_{i=1}^{k} \pi t_i\right]_{m}$ *for all* $\pi \in \mathsf{Perm}(\mathbb{A})$,
4. $\left[\left(\sum_{i=1}^{k} t_i\right)\sigma\right]_{m+l} = \left[\sum_{i=1}^{k} t_i\sigma\right]_{m+l}$ *for all substitutions* $\sigma : X \to \mathsf{Term}_{\Sigma,l}(Y)$.

Proof. We will only show (1) by induction on k, the other statements are analogous.

- *For $k = 0$:* We have

$$\begin{aligned}
\left[a\left(\sum_{i=1}^{k} t_i\right)\right]_{m+1} &= [a\bot]_{m+1} && \text{by definition of } \Sigma\\
&= [\bot]_{m+1} && \text{by Lemma 4.31 (7)}\\
&= \left[\sum_{i=1}^{k} at_i\right]_{m+1} && \text{by definition of } \Sigma.
\end{aligned}$$

- *For $k = k' + 1$:* We have

$$\begin{aligned}
\left[a\left(\sum_{i=1}^{k} t_i\right)\right]_{m+1} &= \left[a\left(\sum_{i=1}^{k'} t_i + t_k\right)\right]_{m+1} && \text{by definition of } \Sigma\\
&= \left[a\left(\sum_{i=1}^{k'} t_i\right) + at_k\right]_{m+1} && \text{by Lemma 4.31 (5)}\\
&= \left[\sum_{i=1}^{k'} at_i + at_k\right]_{m+1} && \text{by inductive hypothesis}\\
&= \left[\sum_{i=1}^{k} at_i\right]_{m+1} && \text{by definition of } \Sigma.
\end{aligned}$$

□

Lemma 4.34 (Properties of Enumerated Sums)**.** *Let* $t \in \mathsf{Term}_{\Sigma,m}(X)$ *with* $t = \sum_{i=1}^{k} t_i$*. Then the following properties hold:*

1. *If* $u \in \mathsf{Term}_{\Sigma,m}(X)$ *and* $[u]_m = [u + t_i]_m$ *for every* $i \in \{1, \ldots, k\}$*, then* $[u]_m = [u + t]_m$.
2. *If* $u \in \mathsf{Term}_{\Sigma,m}(X)$ *and* $[t_i]_m = [t_i + u]_m$ *for some* $i \in \{1, \ldots, k\}$*, then* $[t]_m = [t + u]_m$.
3. *If* $u \in \mathsf{Term}_{\Sigma,m}(X)$ *and* $[t_i]_m = [u]_m$ *for some* $i \in \{1, \ldots, k\}$*, then* $[t]_m = [t + u]_m$.

Proof. We will show each statement individually:

1. By induction on k.
   - *For $k = 0$:* We have $[u]_m = \left[u + \sum_{i=1}^{k} t_i\right]_m = [u + \bot]_m = [u + t]_m$ by Lemma 4.31 (4).
   - *For $k = k' + 1$:* It follows that

$$\begin{aligned}
[u]_m &= [u + t_1 + \ldots + t_{k'}]_m && \text{by inductive hypothesis}\\
&= [(u + t_k) + t_1 + \ldots + t_{k'}]_m && \text{by assumption with } i = k\\
&= [u + t_1 + \ldots + t_{k'} + t_k]_m && \text{by reordering summands}\\
&= [u + t]_m && \text{by assumption.}
\end{aligned}$$

2. By simple computation we get

$$\begin{aligned}
[t]_m &= \left[\sum_{i=1}^{k} t_i\right]_m && \text{by assumption}\\
&= [t_1 + \ldots + t_i + \ldots + t_k]_m && \text{by definition of } \Sigma\\
&= [t_1 + \ldots + (t_i + u) + \ldots + t_k]_m && \text{by assumption}\\
&= [(t_1 + \ldots + t_i + \ldots + t_k) + u]_m && \text{by reordering summands}\\
&= \left[\sum_{i=1}^{k} t_i + u\right]_m && \text{by definition of } \Sigma\\
&= [t + u]_m && \text{by assumption.}
\end{aligned}$$

3. It follows from Lemma 4.31 (3) and the assumption that $[t_i]_m = [t_i + t_i]_m = [t_i + u]_m$. By application of (2), we get the expected result.

□

Lemma 4.35 (Properties of Sums over Sets)**.** *Let $S, S' \in \mathcal{P}_\mathsf{f}(\mathsf{Term}_{\Sigma,m}(X)$ be finite sets of terms. Then the following properties hold:*

1. *If S and S' are derivably equal in that $\{[t]_m : t \in S\} = \{[t']_m : t' \in S'\}$, then $\left[\sum_{t\in S} t\right]_m = \left[\sum_{t'\in S'} t'\right]_m$.*
2. $\left[\sum_{t\in S} t + \sum_{t'\in S'} t'\right]_m = \left[\sum_{u\in S\cup S'} u\right]_m$.

Proof. Let $S = \{t_1, \ldots, t_k\}$ and $S' = \{t'_1, \ldots, t'_{k'}\}$ with $k, k' \in \mathbb{N}_0$ and $t_1, \ldots, t_k, t'_1, \ldots, t'_{k'} \in \mathsf{Term}_{\Sigma,m}(X)$.

1. We will first show that

$$\left[\sum_{t\in S} t\right]_m = \left[\sum_{i=1}^{k} t_i\right]_m = \left[\sum_{i=1}^{k} t_i + \sum_{i=1}^{k'} t'_i\right]_m = \left[\sum_{t\in S} t + \sum_{t'\in S'} t'\right]_m.$$

By Lemma 4.34 (1), it is enough to show $\left[\sum_{i=1}^{k} t_i\right] = \left[\sum_{i=1}^{k} t_i + t'_j\right]$ for every $j \in \{1, \ldots, k'\}$. We know that S and S' are derivably equal, and thus $[t'_j]_m \in \{[t]_m : t \in S\}$. So there exists a $t_l \in S$ with $[t_l]_m = [t'_j]$. The statement then follows from Lemma 4.34 (3).

With a similar argument, one can show that

$$\left[\sum_{t'\in S'} t'\right]_m = \left[\sum_{t'\in S'} t' + \sum_{t\in S} t\right]_m.$$

Thus, equality follows from Lemma 4.31 (1).

2. By definition, $S \cup S' = \{t_1, \ldots, t_k, t'_1, \ldots, t'_{k'}\}$. It then follows that by definition of Σ that

$$\left[\sum_{t\in S} t + \sum_{t'\in S'} t'\right]_m = [t_1 + \ldots + t_k + t'_1 + \ldots + t'_{k'}]_m = \left[\sum_{t\in S\cup S'} t\right]_m.$$

□

Lemma 4.36. *If $S \in \mathcal{P}_\mathsf{f}(\mathcal{P}_\mathsf{f}(\mathsf{Term}_{\Sigma,m}(X)))$ is finite nested set of terms, then*

$$\left[\sum_{L\in S} \sum_{t\in L} t\right] = \left[\sum_{t\in \bigcup S} t\right].$$

Proof. This follows directly from Lemma 4.35 (2) by routine induction on $|S|$. □

Of special importance is the fact that every term is derivably equal to a "flattened" version of it, containing only single pretraces as summands:

Proposition 4.37. *If* $t \in \mathsf{Term}_{m,\Sigma}(X)$ *is a term, then* $[t]_m = \left[\sum_{i=1}^k w_i\right]_m$, *where* $k \in \mathbb{N}_0$ *and* $w_1, \ldots, w_k \in \bar{\mathbb{A}}^m \times X$ *are pretraces.*

Proof. By induction on t.

- *For* $t = x$ *with* $x \in X$*:* Set $k = 1$ and $w_1 = x$. Then we get $\left[\sum_{i=1}^k w_i\right]_0 = [x]_0 = [t]_0$.
- *For* $t = \bot$*:* Set $k = 0$. Set $k = 0$. By definition of Σ, we get $\left[\sum_{i=1}^k w_i\right]_m = [\bot]_m = [t]_m$.
- *For* $t = at'$ *with* $t' \in \mathsf{Term}_{\Sigma,m'}(X), m = m' + 1$*:* By inductive hypothesis, we know that

$$[t']_{m'} = \left[\sum_{i=1}^k w_i\right]_{m'}$$

  with $k \in \mathbb{N}_0$ and $w_1, \ldots, w_k \in \bar{\mathbb{A}}^{m'} \times X$. It then follows from Lemma 4.33 (1) that

$$[at']_m = \left[a\left(\sum_{i=1}^k w_i\right)\right]_m = \left[\sum_{i=1}^k aw_i\right]_m.$$

  Since aw_i is a pretrace for every $i \in \{1, \ldots, k\}$, the statement holds.
- *For* $t = |at'$*:* Analogous to the above case.
- *For* $t = r + s$ *with* $u, v \in \mathsf{Term}_{\Sigma,m}(X)$*:* By inductive hypothesis, we know that

$$[r]_m = \left[\sum_{i=1}^k w_i\right]_m \quad \text{and} \quad [s]_m = \left[\sum_{i=1}^l v_i\right]$$

  with $k, l \in \mathbb{N}_0$ and $w_1, \ldots, w_k, v_1, \ldots, v_l \in \bar{\mathbb{A}}^{m'} \times X$. It then follows that

$$[r + s]_m = \left[\sum_{i=1}^k w_i + \sum_{i=1}^l v_i\right]_m = [w_1 + \ldots + w_k + v_1 + \ldots + v_l]_m$$

  is a sum of pretraces.

□

Proposition 4.38. *If* $k \in \mathbb{N}_0$ *and* $w_1, \ldots, w_k \in \bar{\mathbb{A}}^m \times X$ *are pretraces, then*

$$\Phi_m\left(\left[\sum_{i=1}^k w_i\right]_m\right) = \bigcup_{i=1}^k \hat{D}([w_i]_{\hat{\alpha}}).$$

Proof. By induction on k.

- *For* $k = 0$*:* By definition, we have

$$\Phi_m\left(\left[\sum_{i=1}^k w_i\right]_m\right) = \Phi_m([\bot]_m) = \bot_{F'(X),m} = \emptyset = \bigcup_{i=1}^k \hat{D}([w_i]_{\hat{\alpha}}).$$

- *For $k = k' + 1$:* It follows that

$$\begin{aligned}
&\Phi_m\left(\left[\textstyle\sum_{i=1}^{k} w_i\right]_m\right) && \\
&= \Phi_m\left(\left[\textstyle\sum_{i=1}^{k'} w_i + w_k\right]_m\right) && \text{by definition of } \Sigma \\
&= +_{F'(X),m}\left(\Phi_m\left(\left[\textstyle\sum_{i=1}^{k'} w_i\right]_m\right), \Phi_m([w_k]_m)\right) && \text{by Lemma 4.29} \\
&= \Phi_m\left(\left[\textstyle\sum_{i=1}^{k'} w_i\right]_m\right) \cup \Phi_m([w_k]_m) && \text{by Definition 4.20} \\
&= \textstyle\bigcup_{i=1}^{k'} \hat{D}([w_i]_{\hat{\alpha}}) \cup \Phi_m([w_k]_m) && \text{by inductive hypothesis} \\
&= \textstyle\bigcup_{i=1}^{k'} \hat{D}([w_i]_{\hat{\alpha}}) \cup \hat{D}([w_k]_{\hat{\alpha}}) && \text{by Theorem 4.30} \\
&= \textstyle\bigcup_{i=1}^{k} \hat{D}([w_i]_{\hat{\alpha}}) && \text{by definition.}
\end{aligned}$$

□

Proposition 4.39. *Let $t \in \mathsf{Term}_{\Sigma,m}(X)$ be a term and $w \in \bar{\mathbb{A}}^m \times X$ a pretrace. If $\hat{D}_m([w]_{\hat{\alpha}}) \subseteq \Phi_m([t]_m)$, then $[t]_m = [t + w]_m$.*

Proof. By induction on m.

We know from Proposition 4.37 that $[t]_m = \left[\sum_{i=1}^{k} v_i\right]_m$ for some $k \in \mathbb{N}_0$ and pretraces $v_1, \ldots, v_k \in \bar{\mathbb{A}}^m \times X$.

- *For $w = x \in X$ and $m = 0$:* We know that $v_i \in X$ for all $i \in \{1, \ldots, k\}$.

  It follows from Proposition 4.13 that $\hat{D}([w]_{\hat{\alpha}}) = \{\hat{\mathsf{ub}}(x)\} = \{x\}$. By Proposition 4.38, we also get

$$\Phi_0([t]_0) = \Phi_0\left(\left[\textstyle\sum_{i=1}^{k} v_i\right]_0\right) = \textstyle\bigcup_{i=1}^{k} \hat{D}([v_i]_{\hat{\alpha}}) = \bigcup_{i=1}^{k}\{v_i\} = \{v_1, \ldots, v_p\}.$$

  Since $\hat{D}([w]_{\hat{\alpha}}) \subseteq \Phi_0([t]_0)$, it follows that $x = v_i$ for some $i \in \{1, \ldots, p\}$ and thus also $[x]_0 = [v_i]_0$. The statement then follows from Lemma 4.34 (3).

- *For $w = aw'$ with $w' \in \bar{\mathbb{A}}^{m'} \times X, m = m' + 1$:* We will define a new sum term

$$t' := \sum_{\substack{i=1 \\ v_i = av'_i}}^{k} v'_i + \sum_{\substack{i=1 \\ v_i = \mathsf{l}av'_i}}^{k} v'_i + \sum_{\substack{i=1 \\ v_i = \mathsf{l}bv'_i, a \neq b, a \notin \mathsf{FN}(v'_i)}}^{k} (a\ b)v'_i.$$

  Note that it follows from Lemma 4.33 (1) that

$$[at']_m = \left[\sum_{\substack{i=1 \\ v_i = av'_i}}^{k} av'_i + \sum_{\substack{i=1 \\ v_i = \mathsf{l}av'_i}}^{k} av'_i + \sum_{\substack{i=1 \\ v_i = \mathsf{l}bv'_i, a \neq b, a \notin \mathsf{FN}(v'_i)}}^{k} a((a\ b)v'_i)\right]_m .$$

  Next, we will show that

$$[t]_m = [t + at']_m \tag{4.10}$$

  using Lemma 4.34 (1) for all three sums individually:

  1. *For $v_i = av'_i$:* Then $[v_i]_m = [av'_i]_m$ and, by Lemma 4.34 (3), $[t]_m = [t + av'_i]_m$.

2. *For* $v_i = \mathsf{I}av'_i$*:* It follows from Lemma 4.31 (9) that $[v_i]_m = [v_i + av'_i]_m$. By Lemma 4.34 (2), we have $[t]_m = [t + av'_i]_m$.

3. *For* $v_i = \mathsf{I}bv'_i$ *with* $a \neq b$ *and* $a \notin \mathsf{FN}(v'_i)$*:* It follows from Proposition 4.13 (4) and Lemma 4.4 that $[\mathsf{I}bv'_i]_m = [\mathsf{I}a((a\ b)v'_i)]_m$. We can then conclude from Lemma 4.31 (9) that $[v_i]_m = [v_i + a((a\ b)v'_i)]_m$ and, by Lemma 4.34 (2), $[t]_m = [t + a((a\ b)v'_i)]_m$.

Furthermore, we will show that $\hat{D}([w']_{\hat{\alpha}}) \subseteq \Phi_{m'}([t']_{m'})$. So let $x \in \hat{D}([w']_{\hat{\alpha}})$. By Proposition 4.38, it is enough to show that $x \in \hat{D}([u]_{\hat{\alpha}})$ for some summand u in t'.

It follows from Proposition 4.18 (1) that $ax \in \hat{D}([aw']_{\hat{\alpha}}) \subseteq \Phi_m([t]_m)$. By Proposition 4.38, we then know that $ax \in \hat{D}([v_i]_{\hat{\alpha}})$ for some $i \in \{1, \dots, k\}$. By Proposition 4.18, we only need to consider the following cases for v_i:

- *If* $v_i = av'_i$*:* Then $x \in \hat{D}([v'_i]_{\hat{\alpha}})$ and v'_i is a summand in t'.
- *If* $v_i = \mathsf{I}av'_i$*:* Then $x \in \hat{D}([v'_i]_{\hat{\alpha}})$ and v'_i is a summand in t'.
- *If* $v_i = \mathsf{I}bv'_i$ *with* $a \neq b$*:* We know that $a \notin \mathsf{FN}(v'_i)$ and $x \in \hat{D}([(a\ b)v'_i]_{\hat{\alpha}})$. By definition, $(a\ b)v'_i$ is a summand in t'.

We can conclude that

$$\begin{aligned}
[t]_m &= [t + at']_m && \text{by Equation 4.10} \\
&= [t + a(t' + w')]_m && \text{by inductive hypothesis} \\
&= [t + at' + aw']_m && \text{by Lemma 4.31 (5)} \\
&= [t + aw']_m && \text{by Equation 4.10.}
\end{aligned}$$

- *For* $w = \mathsf{I}aw'$ *with* $w' \in \bar{\mathbb{A}}^{m'} \times X, m = m' + 1$*:* Pick $c \in \mathbb{A} \setminus \mathsf{FN}(w')$ s.t. $c \neq a$ and there is no $i \in \{1, \dots, k\}$ with $v_i = cv'_i$ or $v_i = \mathsf{I}cv'_i$ (so there is no term prefixed with c). This is always possible because there are only k terms and thus only finitely different prefixes used.

  We will now define a new sum term

  $$t' := \sum_{\substack{i=1 \\ v_i = \mathsf{I}bv'_i, c \notin \mathsf{FN}(v'_i)}}^{k} (b\ c)v'_i.$$

  Note that it follows from Lemma 4.33 (2) that

  $$[\mathsf{I}ct']_m = \left[\sum_{\substack{i=1 \\ v_i = \mathsf{I}bv'_i, c \notin \mathsf{FN}(v'_i)}}^{k} \mathsf{I}c((b\ c)v'_i) \right]_m .$$

  Next, we will show that

  $$[t]_m = [t + \mathsf{I}ct']_m \tag{4.11}$$

  using Lemma 4.34 (1): Assume $v_i = \mathsf{I}bv'_i$ with $c \notin \mathsf{FN}(v'_i)$. Since $b \neq c$ by choice of c, it follows from Proposition 4.13 (4) and Lemma 4.4 that $[\mathsf{I}bv'_i]_m = [\mathsf{I}c((b\ c)v'_i)]_m$. We can thus conclude $[t]_m = [t + \mathsf{I}bv'_i]$ from Lemma 4.34 (3).

  Note that, since $c \neq a$ and $c \notin \mathsf{FN}(w')$ by choice of c, it follows from Proposition 4.13 (4) and Lemma 4.4 that

  $$[\mathsf{I}aw']_m = [\mathsf{I}c((a\ c)w')]_m. \tag{4.12}$$

Furthermore, we will show that $\hat{D}([(a\ c)w']_{\hat{\alpha}}) \subseteq \Phi_{m'}([t']_{m'})$. So let $x \in \hat{D}([(a\ c)w']_{\hat{\alpha}})$. By Proposition 4.38, it is enough to show that $x \in \hat{D}([u]_{\hat{\alpha}})$ for some summand u in t'.

It follows from Proposition 4.18 (2) that $cx \in \hat{D}([|c((a\ c)w')]_{\hat{\alpha}}) = \hat{D}([|aw']_{\hat{\alpha}}) \subseteq \Phi_m([t]_m)$. By Proposition 4.38, we then know that $cx \in \hat{D}([v_i]_{\hat{\alpha}})$ for some $i \in \{1, \ldots, k\}$. By choice of c, we only need to consider the case $v_i = |bv_i'$ with $b \neq c$, as there are no pretraces beginning with c or $|c$. It then follows from Proposition 4.18 (3) that $c \notin \mathsf{FN}(v_i')$ and $x \in \hat{D}([(b\ c)v_i']_{\hat{\alpha}})$. By definition, $(b\ c)v_i'$ is a summand in t'.

We can conclude that

$$
\begin{aligned}
[t]_m &= [t + |ct']_m && \text{by Equation 4.11} \\
&= [t + |c(t' + (a\ c)w')]_m && \text{by inductive hypothesis} \\
&= [t + |ct' + |c((a\ c)w')] && \text{by Lemma 4.31 (6)} \\
&= [t + |c((a\ c)w')] && \text{by Equation 4.11} \\
&= [t + |aw'] && \text{by Equation 4.12.}
\end{aligned}
$$

□

This can be used to finally prove injectivity of Φ for all terms:

Theorem 4.40. *The function Φ_m is injective for every $m \in \mathbb{N}_0$.*

Proof. Let $t, u \in \mathsf{Term}_{\Sigma,m}(X)$ with $\Phi_m([t]_m) = \Phi_m([u]_m)$. We have to show $[t]_m = [u]_m$.

We will first show $[t]_m = [t + u]_m$.

It follows from Proposition 4.37 that $[u]_m = \left[\sum_{i=1}^{k} u_i\right]_m$ for some $k \in \mathbb{N}_0$ and pretraces $u_1, \ldots, u_k \in \bar{\mathbb{A}}^m \times X$. By Lemma 4.34 (1), it is enough to show $[t]_m = [t + u_i]_m$ for every $i \in \{1, \ldots, k\}$.

We know that $\hat{D}([u_i]_{\hat{\alpha}}) \subseteq \bigcup_{j=1}^{k} \hat{D}([u_j]_{\hat{\alpha}}) = \Phi_m([u]_m)$, where the last equality follows from Proposition 4.38. By assumption, it then follows that $\hat{D}([u_i]_{\hat{\alpha}}) \subseteq \Phi_m([t]_m)$. Thus, we can conclude $[t]_m = [t + u_i]_m$ from Proposition 4.39.

With a similar argument, one can show $[u]_m = [u + t]_m$. The equality then follows from Lemma 4.31 (1). □

5 Graded Semantics for RNNAs

Now that we have constructed a graded theory to capture local-freshness semantics and shown that this theory induces a graded monad, we turn our attention to actual RNNAs and show that we can use this induced graded monad to capture the local-freshness trace semantics of them.

Definition 5.1. Let the **graded semantics for RNNAs under local freshness** be defined by the graded monad $((M_n), \eta, (\mu^{nk}))$ induced by the graded theory given in Definition 4.1, as well as the natural transformation $\alpha : H \to M_1$ with

$$\alpha_X : \mathcal{P}_{\mathsf{f}}(\mathbb{A} \times X) \times \mathcal{P}_{\mathsf{f}}([\mathbb{A}]X) \to \mathsf{Term}_{\Sigma,1}(X)/{\sim},$$
$$\alpha_X(S_{\mathsf{f}}, S_{\mathsf{b}}) = \left[\sum_{(a,x)\in S_{\mathsf{f}}} ax + \sum_{(a,x)\in u_X[S_{\mathsf{b}}]} \mathsf{l}ax \right]_1,$$

where $u_X : [\mathbb{A}]X \to \mathbb{A}\times X$ is any splitting of $[\mathbb{A}]X$, in that, if $u_X(\langle a\rangle x) = (b, y)$, then $\langle a\rangle x = \langle b\rangle y$.

Lemma 5.2. *The above description of α is indeed a well-defined natural transformation, in that*

1. *The definition of α_X is invariant under the choice of u_X for every $X \in \mathsf{Ob}(\mathsf{Nom})$.*
2. *α_X is equivariant for every $X \in \mathsf{Ob}(\mathsf{Nom})$.*
3. *α is natural.*

Proof. We will first show that, for $a, a' \in \mathbb{A}$ and $x, x' \in X$, we have

$$\langle a\rangle x = \langle a'\rangle x' \quad \Longrightarrow \quad [\mathsf{l}ax]_1 = [\mathsf{l}a'x']_1.$$

By Proposition 2.8 there are two cases to consider:

- If $a = a'$ and $x = x'$, then $[\mathsf{l}ax]_1 = [\mathsf{l}a'x']_1 \in S'$.
- If $a \neq a'$, $a\#x'$, and $x = (a\ a')x'$, then we know $a \notin \mathsf{FN}(x') = \mathsf{supp}(x')$. It then follows from Proposition 4.13 (4) that $[\mathsf{l}ax]_1 = [\mathsf{l}a'x']_1 \in S'$.

With this, we can conclude that, for a set $S_{\mathsf{b}} \in \mathcal{P}([\mathbb{A}]X)$ and any splitting $u_X : [\mathbb{A}]X \to \mathbb{A} \times X$, we have

$$\{[\mathsf{l}ax]_1 : (a,x) \in u_X[S_{\mathsf{b}}]\} = \{[\mathsf{l}ax]_1 : \langle a\rangle x \in S_{\mathsf{b}}\}. \tag{5.1}$$

We will proceed to show the statements individually.

1. Let $u_X, u'_X : [\mathbb{A}]X \to \mathbb{A} \times X$ be two splittings of $[\mathbb{A}]X$.

   By Lemma 3.18, it is enough to show

$$\left[\sum_{(a,x)\in u_X[S_{\mathsf{b}}]} \mathsf{l}ax = \sum_{(a,x)\in u'_X[S_{\mathsf{b}}]} \mathsf{l}ax \right]_1.$$

It follows from Equation 5.1 that

$$\{[|ax]_1 : (a,x) \in u_X[S_\mathsf{b}]\} = \{[|ax]_1 : \langle a\rangle x \in S_\mathsf{b}\} = \{[|ax]_1 : (a,x) \in u'_X[S_\mathsf{b}]\}.$$

The statement then follows directly from Lemma 4.35 (1).

2. Let $\pi \in \mathsf{Perm}(\mathbb{A})$, $S_\mathsf{f} \in \mathcal{P}_\mathsf{f}(\mathbb{A} \times X)$, and $S_\mathsf{b} \in \mathcal{P}_\mathsf{f}([\mathbb{A}]X)$.

   We have to show equality for the equivalence classes of terms

$$\alpha_X(\pi S_\mathsf{f}, \pi S_\mathsf{b}) = \left[\sum_{(a,x)\in \pi S_\mathsf{f}} ax + \sum_{(a,x)\in u_X[\pi S_\mathsf{b}]} |ax\right]_1$$

   and

$$\begin{aligned}
\pi\alpha_X(S_\mathsf{f}, S_\mathsf{b}) &= \pi\left[\sum_{(a,x)\in S_\mathsf{f}} ax + \sum_{(a,x)\in u_X[S_\mathsf{b}]} |ax\right]_1 \\
&= \left[\pi\left(\sum_{(a,x)\in S_\mathsf{f}} ax + \sum_{(a,x)\in u_X[S_\mathsf{b}]} |ax\right)\right]_1 && \text{by Proposition 2.12} \\
&= \left[\sum_{(a,x)\in S_\mathsf{f}} \pi(ax) + \sum_{(a,x)\in u_X[S_\mathsf{b}]} \pi(|ax)\right]_1 && \text{by Lemma 4.33 (3).}
\end{aligned}$$

   By Lemma 4.35 (1), it is then enough to show equality of the summands in both both sums: For the first sum, we have

$$\{[(\pi a)(\pi x)]_1 : (\pi a, \pi x) \in \pi S_\mathsf{f}\} = \{[(\pi a)(\pi x)]_1 : (a,x) \in S_\mathsf{f}\} = \{[\pi(ax)]_1 : (a,x) \in S_\mathsf{f}\}.$$

   For the second sum, we have

$$\begin{aligned}
\{[|ax]_1 : (a,x) \in u_X[\pi S_\mathsf{b}]\} &= \{[|(\pi a)(\pi x)]_1 : \langle \pi a, \pi x\rangle \in \pi S_\mathsf{b}\} && \text{by Equation 5.1} \\
&= \{[|(\pi a)(\pi x)]_1 : \langle a\rangle x \in S_\mathsf{b}\} && \text{by definition} \\
&= \{[\pi(|ax)]_1 : \langle a\rangle x \in S_\mathsf{b}\} && \text{by Definition 3.5} \\
&= \{[\pi(|ax)]_1 : (a,x) \in u_X[S_\mathsf{b}]\} && \text{by Equation 5.1.}
\end{aligned}$$

3. Let $X, Y \in \mathsf{Ob}(\mathsf{Nom})$ and $f : X \to Y$ equivariant.

   We have to show that, for every $F \in \mathcal{P}_\mathsf{f}(\mathbb{A} \times X)$ and $B \in \mathcal{P}_\mathsf{f}([\mathbb{A}]X)$, $M_1(f)(\alpha_X(S_\mathsf{f}, S_\mathsf{b})) = \alpha_Y(H(f)(S_\mathsf{f}, S_\mathsf{b}))$.

   By explicit computation, we get

$$\begin{aligned}
M_1(f)(\alpha_X(S_\mathsf{f}, S_\mathsf{b})) &= M_1(f)\left(\left[\sum_{(a,x)\in S_\mathsf{f}} ax + \sum_{(a,x)\in u_X[S_\mathsf{b}]} |ax\right]_1\right) && \text{by definition of } \alpha \\
&= \left[\left(\sum_{(a,x)\in S_\mathsf{f}} ax + \sum_{(a,x)\in u_X[S_\mathsf{b}]} |ax\right)\sigma_f\right]_1 && \text{by Definition 3.28} \\
&= \left[\sum_{(a,x)\in S_\mathsf{f}} (ax)\sigma_f + \sum_{(a,x)\in u_X[S_\mathsf{b}]} (|ax)\sigma_f\right]_1 && \text{by Lemma 4.33 (4),}
\end{aligned}$$

where σ_f is defined as in Definition 3.25. On the other hand,

$$\begin{aligned}\alpha_Y(H(f)(S_{\mathsf{f}}, S_{\mathsf{b}})) &= \alpha_Y(\{(a, f(x)) : (a,x) \in S_{\mathsf{f}}\}, \underbrace{\{\langle a\rangle f(x) : \langle a\rangle x \in S_{\mathsf{b}}\}}_{=:S'_{\mathsf{b}}}) \\ &= \left[\sum_{(a,x)\in S_{\mathsf{f}}} a(f(x)) + \sum_{(a,x)\in u[S'_{\mathsf{b}}]} |ax\right]_1 .\end{aligned}$$

Once again, by Lemma 4.35 (1), we only need to show equality of the summands in both sums. By Definition 3.4, we get

$$\{[(ax)\sigma_f]_1 : (a,x) \in S_{\mathsf{f}}\} = \{[a(\sigma_f(x))]_1 : (a,x) \in S_{\mathsf{f}}\} = \{[a(f(x))]_1 : (a,x) \in S_{\mathsf{f}}\},$$

and for the second sum

$$\begin{aligned}\{[(|ax)\sigma_f]_1 : (a,x) \in S_{\mathsf{b}}\} &= \{[(|ax)\sigma_f]_1 : \langle a\rangle x \in S_{\mathsf{b}}\} && \text{by Equation 5.1} \\ &= \{[|a(f(x))]_1 : \langle a\rangle x \in S_{\mathsf{b}}\} && \text{by Definition 3.4} \\ &= \{[|a(x)]_1 : \langle a\rangle x \in S'_{\mathsf{b}}\} && \text{by definition of } S'_{\mathsf{b}} \\ &= \{[|a(x)]_1 : (a,x) \in u[S'_{\mathsf{b}}]\} && \text{by Equation 5.1.}\end{aligned}$$

□

5.1 Capturing the Bar Language

We will proceed to show that the terms produced by the graded semantics capture the bar language of the RNNA.

First, we will introduce some lemmas to simplify evaluation of the graded monad multiplication.

Lemma 5.3. *If $w[t]_k \in \bar{\mathbb{A}}^n \times M_k(X)$ is a pretrace over equivalence classes of terms, then $\mu_X^{nk}([w[t]_k]_n) = [wt]_{n+k}$.*

Proof. By induction on n.

- *For $w = \varepsilon$ and $n = 0$:*

$$\begin{aligned}\mu_X^{0k}([[t]_k]_n) &= [\![[t]_k]\!]_0^{\mathsf{id}} && \text{by Definition 3.28} \\ &= [t]_k && \text{by Definition 3.13.}\end{aligned}$$

- *For $w = aw'$ with $w' \in \bar{\mathbb{A}}^{n'}$, $n = n' + 1$:*

$$\begin{aligned}\mu_X^{nk}\left([aw'[t]_k]_n\right) &= [\![aw'[t]_k]\!]_n^{\mathsf{id}} && \text{by Definition 3.28} \\ &= \mathsf{pre}_{F(X),n'+k}(a, [\![w'[t]_k]\!]_{n'}^{\mathsf{id}}) && \text{by Definition 3.13} \\ &= \mathsf{pre}_{F(X),n'+k}\left(a, \mu_X^{n'k}\left([w'[t]_k]_{n'}\right)\right) && \text{by Definition 3.28} \\ &= \mathsf{pre}_{F(X),n'+k}(a, [w't]_{n'+k}) && \text{by inductive hypothesis} \\ &= [aw't]_{n+k} && \text{by Definition 3.19.}\end{aligned}$$

- *For $w = |aw'$:* Analogous to the above case.

□

Lemma 5.4. *If $l \in \mathbb{N}_0$ and $w_1[t_1]_k, \ldots, w_l[t_l]_k \in \bar{\mathbb{A}}^n \times M_k(X)$ are pretraces over equivalence classes of terms, then*

$$\mu_X^{nk}\left(\left[\sum_{i=1}^{l} w_i[t_i]_k\right]_n\right) = \left[\sum_{i=0}^{l} w_i t_i\right]_{n+k}.$$

Proof. By induction on l.

- *For $l = 0$:*

$$\begin{aligned}
\mu_X^{nk}\left(\left[\sum_{i=1}^{l} w_i[t_i]_k\right]_n\right) &= \mu_X^{nk}([\bot]_n) && \text{by definition of } \Sigma \\
&= [\![\bot]\!]_n^{\mathsf{id}} && \text{by Definition 3.28} \\
&= \bot_{F(X),n+k} && \text{by Definition 3.13} \\
&= [\bot]_{n+k} && \text{by Definition 3.19.}
\end{aligned}$$

- *For $l = l' + 1$:*

$$\begin{aligned}
&\mu_X^{nk}\left(\left[\sum_{i=1}^{l} w_i[t_i]_k\right]_n\right) \\
&= \mu_X^{nk}\left(\left[\sum_{i=1}^{l'} w_i[t_i]_k + w_l[t_l]_k\right]_n\right) && \text{by definition of } \Sigma \\
&= \left[\!\left[\sum_{i=1}^{l'} w_i[t_i]_k + w_l[t_l]_k\right]\!\right]_n^{\mathsf{id}} && \text{by Definition 3.28} \\
&= +_{F(X),n+k}\left(\left[\!\left[\sum_{i=1}^{l'} w_i[t_i]_k\right]\!\right]_n^{\mathsf{id}}, [\![w_l[t_l]_k]\!]_n^{\mathsf{id}}\right) && \text{by Definition 3.13} \\
&= +_{F(X),n+k}\left(\mu_X^{nk}\left(\left[\sum_{i=1}^{l'} w_i[t_i]_k\right]_n\right), \mu_X^{nk}([w_l[t_l]_k]_n)\right) && \text{by Definition 3.28} \\
&= +_{F(X),n+k}\left(\mu_X^{nk}\left(\left[\sum_{i=1}^{l'} w_i[t_i]_k\right]_n\right), [w_l t_l]_{n+k}\right) && \text{by Lemma 5.3} \\
&= +_{F(X),n+k}\left(\left[\sum_{i=1}^{l'} w_i t_i\right]_{n+k}, [w_l t_l]_{n+k}\right) && \text{by inductive hypothesis} \\
&= \left[\sum_{i=1}^{l'} w_i t_i + w_l t_l\right]_{n+k} && \text{by Definition 3.19} \\
&= \left[\sum_{i=1}^{l} w_i t_i\right]_{n+k} && \text{by definition of } \Sigma.
\end{aligned}$$

□

We will continue by characterizing the pretraces of RNNAs.

Definition 5.5 (Pretraces of RNNAs)**.** The **pretraces** generated by a state $s \in Q$ of an RNNA defined by $\gamma : Q \to H(Q)$ are given as

$$\hat{L}_\alpha(s) = \{[wq]_{\hat{\alpha}} : s \xrightarrow{w} q, w \in \bar{\mathbb{A}}^\star, q \in Q\}.$$

We also define the pretraces of length n as $\hat{L}_\alpha^{(n)}(s) = \{[wq]_{\hat{\alpha}} \in \hat{L}_\alpha(s) : |w| = n\}$.

Lemma 5.6. *The pretrace map $\hat{L}_\alpha^{(n)} : Q \to (\bar{\mathbb{A}}^n \times Q)/\hat{\equiv}_\alpha$ is equivariant for all $n \in \mathbb{N}_0$.*

Proof. Let $\pi \in \mathsf{Perm}(\mathbb{A})$. It then follows that

$$\begin{aligned}
\hat{L}^{(n)}_{\alpha}(\pi s) &= \{[(\pi w)(\pi q))]_{\hat{\alpha}} : \pi s \xrightarrow{\pi w} \pi q, \pi w \in \bar{\mathbb{A}}^n, \pi q \in Q\} && \text{by definition} \\
&= \{[(\pi w)(\pi q)]_{\hat{\alpha}} : s \xrightarrow{w} q, w \in \bar{\mathbb{A}}^n, q \in Q\} && \text{by equivariance of } \to \\
&= \{\pi[wq]_{\hat{\alpha}} : s \xrightarrow{w} q, w \in \bar{\mathbb{A}}^n, q \in Q\} && \text{by Proposition 2.12} \\
&= \pi\hat{L}^{(n)}_{\alpha}(s) && \text{by definition.}
\end{aligned}$$

□

The support of a pretrace is limited by the support of the origin state, as shown below:

Lemma 5.7. *Let $s \in Q$ be a state in an RNNA defined by $\gamma : Q \to H(Q)$. If $[wq]_{\hat{\alpha}} \in \hat{L}_{\alpha}(s)$, then $\mathsf{supp}([wq]_{\hat{\alpha}}) = \mathsf{FN}(wq) \subseteq \mathsf{supp}(s)$.*

Proof. By induction on w.

- *For $w = \varepsilon$ and $n = 0$:* We have $s = q$ and, by Definition 4.9, $\mathsf{FN}(q) = \mathsf{FN}(s) = \mathsf{supp}(s)$.
- *For $w = aw'$ with $w' \in \bar{\mathbb{A}}^{n'}$, $n = n' + 1$:* Then we have $s \xrightarrow{a} s' \xrightarrow{w'} q$ for some $s' \in Q$. It follows that

$$\begin{aligned}
\mathsf{FN}(aw'q) &= \mathsf{FN}(w'q) \cup \{a\} && \text{by Definition 4.9} \\
&\subseteq \mathsf{supp}(s') \cup \{a\} && \text{by inductive hypothesis} \\
&\subseteq \mathsf{supp}(s) && \text{by Proposition 2.22.}
\end{aligned}$$

- *For $w = \mathsf{I}aw'$ with $w' \in \bar{\mathbb{A}}^{n'}$, $n = n' + 1$:* Then we have $s \xrightarrow{\mathsf{I}a} s' \xrightarrow{w'} q$ for some $s' \in Q$. It follows that

$$\begin{aligned}
\mathsf{FN}(\mathsf{I}aw'q) &= \mathsf{FN}(w'q) \setminus \{a\} && \text{by Definition 4.9} \\
&\subseteq \mathsf{supp}(s') \setminus \{a\} && \text{by inductive hypothesis} \\
&\subseteq \mathsf{supp}(s) && \text{by Proposition 2.22.}
\end{aligned}$$

□

We can show that the α-pretrace sequence generated by our graded semantics indeed captures these pretraces.

Theorem 5.8. *Let $s \in Q$ be a state in an RNNA defined by $\gamma : Q \to H(Q)$. Then*

$$\gamma^{(n)}(s) = \left[\sum\nolimits_{wq \in v_n\left[\hat{L}^{(n)}_{\alpha}(s)\right]} wq\right]_n,$$

where $v_n : (\bar{\mathbb{A}}^n \times Q)/\hat{\equiv}_{\alpha} \to \bar{\mathbb{A}}^n \times Q$ is any splitting of pretraces, in that $[\cdot]_{\hat{\alpha}} \circ v_n = \mathsf{id}$.

Proof. By induction on n.

- *For $n = 0$:* By definition, we have $\gamma^{(0)}(s) = \eta_Q(s) = [s]_0$. On the other hand, $\to^0$ is just the identity relation, and thus $\hat{L}^{(0)}_{\alpha}(s) = \{[s]_{\hat{\alpha}}\}$. Since, by Proposition 4.13, $[s]_{\hat{\alpha}} = \{s\}$, we know that $v_0[\hat{L}^{(0)}_{\alpha}(s)] = \{s\}$.

- *For $n = n'+1$:* Let $\gamma(s) = (S_\mathsf{f}, S_\mathsf{b})$. Fix a splitting $u_X : [\mathbb{A}]X \to (\mathbb{A}, X)$. By computation, we get

$$
\begin{aligned}
& \gamma^{(n)}(s) \\
= \quad & \text{(by definition of } \gamma^{(n)}) \\
& \mu_Q^{1n}(M_1(\gamma^{(n')})(\alpha_Q(\gamma(s)))) \\
= \quad & \text{(by Definition 5.1)} \\
& \mu_Q^{1n}\left(M_1(\gamma^{(n')})\left(\left[\sum_{(a,q)\in S_\mathsf{f}} aq + \sum_{(a,q)\in u_X[S_\mathsf{b}]} \mathsf{l}aq\right]_1\right)\right) \\
= \quad & \text{(by Definition 3.28)} \\
& \mu_Q^{1n}\left(\left[\left(\sum_{(a,q)\in S_\mathsf{f}} aq + \sum_{(a,q)\in u_X[S_\mathsf{b}]} \mathsf{l}aq\right)\sigma_{\gamma^{(n')}}\right]_1\right) \\
= \quad & \text{(by Lemma 4.33 (4))} \\
& \mu_Q^{1n}\left(\left[\sum_{(a,q)\in S_\mathsf{f}} (aq)\sigma_{\gamma^{(n')}} + \sum_{(a,q)\in u_X[S_\mathsf{b}]} (\mathsf{l}aq)\sigma_{\gamma^{(n')}}\right]_1\right) \\
= \quad & \text{(by Definition 3.4)} \\
& \mu_Q^{1n}\left(\left[\sum_{(a,q)\in S_\mathsf{f}} a(\gamma^{(n')}(q)) + \sum_{(a,q)\in u_X[S_\mathsf{b}]} \mathsf{l}a(\gamma^{(n')}(q))\right]_1\right) \\
= \quad & \text{(by inductive hypothesis)} \\
& \mu_Q^{1n}\left(\left[\sum_{(a,q)\in S_\mathsf{f}} a\left[\sum_{wr\in v_{n'}[\hat{L}_\alpha^{(n')}(q)]} wr\right]_{n'} + \sum_{(a,q)\in u_X[S_\mathsf{b}]} \mathsf{l}a\left[\sum_{wr\in v_{n'}[\hat{L}_\alpha^{(n')}(q)]} wr\right]_{n'}\right]_1\right) \\
= \quad & \text{(by Lemma 5.4)} \\
& \left[\sum_{(a,q)\in S_\mathsf{f}} a\left(\sum_{wr\in v_{n'}[\hat{L}_\alpha^{(n')}(q)]} wr\right) + \sum_{(a,q)\in u_X[S_\mathsf{b}]} \mathsf{l}a\left(\sum_{wr\in v_{n'}[\hat{L}_\alpha^{(n')}(q)]} wr\right)\right]_n \\
= \quad & \text{(by Lemma 4.33 (1) and (2))} \\
& \left[\sum_{(a,q)\in S_\mathsf{f}} \sum_{wr\in v_{n'}[\hat{L}_\alpha^{(n')}(q)]} awr + \sum_{(a,q)\in u_X[S_\mathsf{b}]} \sum_{wr\in v_{n'}[\hat{L}_\alpha^{(n')}(q)]} \mathsf{l}awr\right]_n \\
= \quad & \text{(by Lemma 4.36)} \\
& \left[\sum_{(a,q)\in S_\mathsf{f}, wr\in v_{n'}[\hat{L}_\alpha^{(n')}(q)]} awr + \sum_{(a,q)\in u_X[S_\mathsf{b}], wr\in v_{n'}[\hat{L}_\alpha^{(n')}(q)]} \mathsf{l}awr\right]_n .
\end{aligned}
$$

We will now show that this matches the given description. By Lemma 4.35 (1) and (2), it is enough to show that

$$
\begin{aligned}
S &\coloneqq \{[awr]_n : (a,q) \in S_\mathsf{f}, wr \in v_{n'}[\hat{L}_\alpha^{(n')}(q)]\} \cup \{[\mathsf{l}awr]_n : (a,q) \in u_X[S_\mathsf{b}], wr \in v_{n'}[\hat{L}_\alpha^{(n')}(q)]\} \\
&= \{[wr]_{\hat{\alpha}} : wr \in v_n[\hat{L}_\alpha^{(n)}(s)]\} = \hat{L}_\alpha^{(n)}(s).
\end{aligned}
$$

First, assume that $(a,q) \in S_{\mathsf{f}}$ and $wr \in v_{n'}[\hat{L}_\alpha^{(n')}(q)]$. It follows that $[wr]_{\hat\alpha} \in \hat{L}_\alpha^{(n')}(q)$. By definition of S_{f} and $\hat{L}_\alpha^{(n')}$, we have $s \xrightarrow{a} q \xrightarrow{w} r$ and thus $[awr]_{\hat\alpha} \in \hat{L}_\alpha^{(n)}(s)$.

Now, assume that $(a,q) \in u_X[S_{\mathsf{b}}]$ and $wr \in v_{n'}[\hat{L}_\alpha^{(n')}(q)]$. It follows that $\langle a\rangle q \in S_{\mathsf{b}}$ and $[wr]_{\hat\alpha} \in \hat{L}_\alpha^{(n')}(q)$. By definition of S_{b} and $\hat{L}_\alpha^{(n')}$, we have $s \xrightarrow{|a} q \xrightarrow{w} r$ and thus $[|awr]_n \in \hat{L}_\alpha^{(n)}(s)$.

Conversely, assume $wr \in v_n[\hat{L}_\alpha^{(n)}(s)]$.

If $w = aw'$ for $a \in \mathbb{A}$ and $w' \in \bar{\mathbb{A}}^{n'}$, then there exists some $q \in Q$ with $s \xrightarrow{a} q \xrightarrow{w'} r$ and thus $(a,q) \in S_{\mathsf{f}}$ with $[w'r]_{\hat\alpha} \in \hat{L}_\alpha^{(n')}(q)$. So there is an $x \in v_{n'}[\hat{L}_\alpha^{(n')}(s)]$ with $[x]_{\hat\alpha} = [w'r]_{\hat\alpha}$. It follows from Proposition 4.13 (2) that $[aw'r]_{\hat\alpha} = [ax]_{\hat\alpha} \in S$.

If $w = |aw'$ for $a \in \mathbb{A}$ and $w' \in \bar{\mathbb{A}}^{n'}$, then there exists some $q \in Q$ with $s \xrightarrow{|a} q \xrightarrow{w'} r$ and thus $\langle a\rangle q \in S_{\mathsf{b}}$ with $[w'r]_{\hat\alpha} \in \hat{L}_\alpha^{(n')}(q)$. Then $u_X(\langle a\rangle q) = (a',q') \in u_X[S_{\mathsf{b}}]$ with $\langle a\rangle q = \langle a'\rangle q'$. By Proposition 2.8, we need to consider two cases:

- *If $a = a'$ and $q = q'$:* Then $[|aw'r]_{\hat\alpha} \in S$ follows similar to the above case.
- *If $a \neq a'$, $a' \# q$, and $q' = (a\ a')q$:* By Lemma 5.7, $a' \notin \mathsf{FN}(w'r) \subseteq \mathsf{supp}(q)$. By Lemma 5.6, we get $[(a\ a')(w'r)]_{\hat\alpha} = (a\ a')[w'r]_{\hat\alpha} \in \hat{L}_\alpha^{(n')}((a\ a')q) = \hat{L}_\alpha^{(n')}(q')$. So there exists an $x \in v_n[\hat{L}_\alpha^{(n)}(q')]$ with $[x]_{\hat\alpha} = [(a\ a')(w'r)]_{\hat\alpha}$. Finally, it follows from Proposition 4.13 (4) and (3) that $[|aw'r]_{\hat\alpha} = [|a'((a\ a')(w'r))]_{\hat\alpha} = [|a'x]_{\hat\alpha} \in S$.

□

5.2 Capturing Local Freshness Semantics

To show the correctness of our graded semantics, we still need to show that our definition of alpha-equivalence on pretraces over 1 and alpha-equivalence on bar strings are the same.

Proposition 5.9. *Let $X = 1 = \{\star\}$ and $w, v \in \bar{\mathbb{A}}^n$ be bar strings. Then $(w,\star) \hat{\equiv}_\alpha (v,\star)$ iff $w \equiv_\alpha v$.*

Proof. 'If': Assume that $w \equiv_\alpha v$. Since $\hat{\equiv}_\alpha$ is an equivalence, we only need to consider the case $w = u|ax$ and $v = u|a'x'$ with $\langle a\rangle x = \langle a'\rangle x'$ in $[\mathbb{A}]\bar{\mathbb{A}}^\star$, where $u \in \bar{\mathbb{A}}^m$ and $x, x' \in \bar{\mathbb{A}}^k$ with $n = m+1+k$.

It follows from Proposition 2.8 that there are only two cases:

- *If $a = a'$ and $x = x'$:* Then we have $w = u|ax = u|a'x' = v$ and, because $\hat{\equiv}_\alpha$ is an equivalence, $(w,\star) \hat{\equiv}_\alpha (v,\star)$.
- *If $a \neq a'$, $a\#x'$, and $x = (a\ a')x'$:* Since $\hat{\equiv}_\alpha$ is an equivalence, we have $(x,\star) \hat{\equiv}_\alpha ((a\ a')x',\star)$. Then, since $a \notin \mathsf{FN}((x',\star)) \subseteq \mathsf{supp}(x')$, it follows from Proposition 4.13 (4) that $(|ax,\star) \hat{\equiv}_\alpha (|a'x',\star)$.

  By applying Proposition 4.13 (2) or (3) m times respectively, we get

$$(w,\star) = (u|ax,\star) \hat{\equiv}_\alpha (u|a'x',\star) = (v,\star).$$

'Only if:' We will proceed by induction on m to show a stronger statement: If $(w,\star) \hat{\equiv}_\alpha (v,\star)$, then $uw \equiv_\alpha uv$ for every $u \in \bar{\mathbb{A}}^\star$ (especially for $u = \varepsilon$).

- *For $m = 0$:* We have $w = v = \varepsilon$ and, because $\equiv_\alpha$ is an equivalence, $uw = u \equiv_\alpha u = uv$.

- *For $m = m' + 1$:* It follows from Proposition 4.13 that we only need to consider three cases:
  - If $w = aw'$ and $v = av'$ with $(w', \star) \hat{\equiv}_\alpha (v', \star)$: We set $\tilde{u} = ua$ and, by applying the inductive hypothesis with $\tilde{u}$, we get $uw = uaw' = \tilde{u}w' \equiv_\alpha \tilde{u}v' = uav' = uv$.
  - If $w = |aw'$ and $v = |av'$ with $(w', \star) \hat{\equiv}_\alpha (v', \star)$: We set $\tilde{u} = u|a$ and, by applying the inductive hypothesis with $\tilde{u}$, we get $uw = u|aw' = \tilde{u}w' \equiv_\alpha \tilde{u}v' = u|av' = uv$.
  - If $w = |aw'$ and $v = |bv'$ with $a \neq b$, $a \notin \mathsf{FN}((v', \star))$, and $(w', \star) \hat{\equiv}_\alpha ((a\ b)v', \star)$: It follows from Lemma 4.12 with $N = \{a\}$ that there exists an $\tilde{v}' \in \bar{\mathbb{A}}^{m'}$ with $(v', \star) \hat{\equiv}_\alpha (\tilde{v}', \star)$ and $a \# \tilde{v}'$. Note that, by Proposition 2.8, we have

$$\langle a \rangle (a\ b) \tilde{v}' = \langle b \rangle \tilde{v}'. \tag{5.2}$$

By equivariance of $\hat{\equiv}_\alpha$, we get $(w', \star) \hat{\equiv}_\alpha ((a\ b)\tilde{v}', \star)$. It now follows that

$$\begin{aligned} uw &= u|aw' && \text{by assumption} \\ &\equiv_\alpha u|a((a\ b)\tilde{v}') && \text{by inductive hypothesis with } \tilde{u} = u|a \\ &\equiv_\alpha u|b\tilde{v}' && \text{by definition of } \equiv_\alpha \text{ and Equation 5.2} \\ &\equiv_\alpha u|bv' && \text{by inductive hypothesis with } \tilde{u} = u|b \\ &= uv && \text{by assumption.} \end{aligned}$$

□

With this, we can show that the α-trace sequence generated by our graded semantics captures the local freshness semantics of RNNAs.

Theorem 5.10. *Let $s \in Q$ be a state in an RNNA defined by $\gamma : Q \to H(Q)$ and $n \in \mathbb{N}_0$. Then*

$$\Phi_n(M_n(!)(\gamma^{(n)}(s))) = \{(w, \star) : w \in D(L_\alpha(s)), |w| = n\}.$$

Proof. We fix a splitting $v_n : (\bar{\mathbb{A}}^n \times Q)/\hat{\equiv}_\alpha \to \bar{\mathbb{A}}^n \times Q$.

By computation, we get

$$\begin{aligned} M_n(!)(\gamma^{(n)}(s)) &= M_n(!)\left(\left[\textstyle\sum_{wq \in v_n[\hat{L}_\alpha^{(n)}(s)]} wq\right]_n\right) && \text{by Theorem 5.8} \\ &= \left[\left(\textstyle\sum_{wq \in v_n[\hat{L}_\alpha^{(n)}(s)]} wq\right) \sigma_!\right]_n && \text{by Definition 3.28} \\ &= \left[\textstyle\sum_{wq \in v_n[\hat{L}_\alpha^{(n)}(s)]} (wq)\sigma_!\right]_n && \text{by Lemma 4.33 (4)} \\ &= \left[\textstyle\sum_{wq \in v_n[\hat{L}_\alpha^{(n)}(s)]} w(!(q))\right]_n && \text{by Definition 3.4} \\ &= \left[\textstyle\sum_{wq \in v_n[\hat{L}_\alpha^{(n)}(s)]} w\star\right]_n && \text{by definition of !.} \end{aligned}$$

It then follows that

$$\begin{aligned} \Phi_n(M_n(!)(\gamma^{(n)}(s))) &= \bigcup_{wq \in v_n[\hat{L}_\alpha^{(n)}(s)]} \hat{D}([(w, \star)]_{\hat{\alpha}}) && \text{by Proposition 4.38} \\ &= \{(x, \star) \in \hat{D}([(w, \star)]_{\hat{\alpha}}) : wq \in v_n[\hat{L}_\alpha^{(n)}(s)]\}. \end{aligned}$$

So assume $(x, \star) \in \hat{D}([(w, \star)]_{\hat{\alpha}})$ with $wq \in v_n[\hat{L}_\alpha^{(n)}(s)]$. It follows that $[wq]_{\hat{\alpha}} \in \hat{L}_\alpha^{(n)}(s)$. By definition of $\hat{L}_\alpha^{(n)}$, we know that $s \xrightarrow{w'} q'$ with $[w'q']_{\hat{\alpha}} = [wq]_{\hat{\alpha}}$ and thus $[w']_\alpha \in L_\alpha(s)$. It furthermore follows from Proposition 4.16 that $[(w', \star)]_{\hat{\alpha}} = [(w, \star)]_{\hat{\alpha}}$ and, by Proposition 5.9, $[w]_\alpha = [w']_\alpha \in L_\alpha(s)$.

Since $(x, \star) \in \hat{D}([(w, \star)]_{\hat{\alpha}})$, we also have $(x, \star) = \hat{\mathsf{ub}}((\tilde{x}, \star))$ for some $(\tilde{x}, \star) \in [(w, \star)]_{\hat{\alpha}}$ and thus $x = \mathsf{ub}(\tilde{x})$. It follows from Proposition 5.9 that $[\tilde{x}]_\alpha = [w]_\alpha$. By Definition 2.20, we can conclude that $x = \mathsf{ub}(\tilde{x}) \in D(L_\alpha(s))$.

Conversely, assume $w \in D(L_\alpha(s))$ with $|w| = n$. By Definition 2.20, we know that $w = \mathsf{ub}(\tilde{w})$ for some $\tilde{w} \in L_\alpha(s)$. So there exists a state $q \in Q$ with $s \xrightarrow{\tilde{w}} q$. It follows that $[\tilde{w}q]_{\hat{\alpha}} \in \hat{L}_\alpha^{(n)}(s)$, and thus $w'q' \in v_n[L^{(n)}(s)]$ with $[w'q']_{\hat{\alpha}} = [\tilde{w}q]_{\hat{\alpha}}$. It follows from Proposition 4.16 that $[(w', \star)]_{\hat{\alpha}} = [(\tilde{w}, \star)]_{\hat{\alpha}}$. We can then conclude that $(w, \star) = \hat{\mathsf{ub}}((\tilde{w}, \star)) \in \hat{D}([(w', \star)]_{\hat{\alpha}})$. □

Corollary 5.11. *Let $q \in Q$ and $s \in S$ be states in the RNNAs defined by $\gamma : Q \to H(Q)$ and $\delta : S \to H(S)$. The states q and s are α-trace equivalent in the graded semantics iff $D(L_\alpha(q)) = D(L_\alpha(s))$.*

Proof. First, we note that

$$D(L_\alpha(x)) = \bigcup_{n \in \mathbb{N}_0} \{w \in D(L_\alpha(x)) : |w| = n\}, \tag{5.3}$$

where the individual subsets are disjoint.

It then follows that

$$\begin{aligned}
& q \text{ and } s\ \alpha\text{-trace equivalent} && \\
\Leftrightarrow\quad & \forall n \in \mathbb{N}_0.\ M_n(!)(\gamma^{(n)}(q)) = M_n(!)(\delta^{(n)}(s)) && \text{by definition} \\
\Leftrightarrow\quad & \forall n \in \mathbb{N}_0.\ \Phi_n(M_n(!)(\gamma^{(n)}(q))) = \Phi_n(M_n(!)(\delta^{(n)}(s))) && \text{by Theorem 4.40} \\
\Leftrightarrow\quad & \forall n \in \mathbb{N}_0.\ \{w \in D(L_\alpha(q)) : |w| = n\} = \{w \in D(L_\alpha(s)) : |w| = n\} && \text{by Theorem 5.10} \\
\Leftrightarrow\quad & D(L_\alpha(q)) = D(L_\alpha(s)) && \text{by Equation 5.3.}
\end{aligned}$$

□

6 Conclusion

In this work, we have demonstrated how the trace semantics of RNNAs under local freshness can be described in the framework of graded monads. To achieve this, we have first introduced a notion of graded theories on the category of nominal sets. This falls in line with similar work that has been done for other categories [For+25; FMS20]. By first defining nominal graded theories in a more general setting and showing that, for every such graded theory, a graded monad on Nom arises from an adjunction (as shown in Corollary 3.29), we have built a framework which could be used or extended to express other semantics.

The actual graded theory for the local freshness semantics was then constructed by first defining a suitable semantics for a single pretraces of an RNNA. By defining alpha-equivalence on pretraces, taking into account the support of the poststate, we were able to show that the interpretation is injective from the equivalence classes of pretraces modulo alpha-equivalence (see Proposition 4.19). This interpretation was then extended to sum terms, which are able to capture all of the alternative pretraces that can be produced by a state in an RNNA, using the fact that RNNAs are finitely branching up to alpha-equivalence. We were also able to show that this extended interpretation is also injective (see Theorem 4.40). Finally, we have shown that by "forgetting" the poststates of those pretraces, we get exactly the traces under local freshness semantics (see Proposition 5.9). With this, we were able to conclude that our graded semantics captures exactly the local freshness trace semantics of RNNAs (see Corollary 5.11).

Furthermore, since we have given an explicit algebraic description of the graded theory in which all operations and equations are at most depth-1, it follows from Theorem 3.36 that the induced graded monad is also depth-1. Once again, by showing the depth-1 property for graded monads induced by arbitrary depth-1 graded theories, this could be used as a foundation for other depth-1 graded theories as well.

Although we have defined the interpretation of pretrace terms as a model $F'(X)$ in Definition 4.20, this definition does not yield a functor in the expected way, as seen in Example 4.27. A possible next step is to give an alternative definition of a functor $\mathsf{Nom} \to \mathsf{Alg}(T, n)$ for our graded theory and thus give an alternative definition of the graded monad, similar to the one seen in Example 2.25 for the trace semantics of LTS. This would also be useful for extending the definition of the functor H to use ufs sets as in the original work instead of finite sets.

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